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A shape optimization algorithm for an elliptic system

by

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This paper deals with the theory of shape optimization (or domain optimization). A model problem for maximization of a prismatic bar torsional rigidity is considered. An algorithm for computation of a generalized solution of the problem is given. The algorithm is based on reduction of the shape optimization problem to a family of coefficient optimization problems for an elliptic equation.

1. Introduction

In the present paper a problem relating to the theory of optimal design of the distributed parameter systems, Banichuk (1983), Pironneau (1984) is considered. It is required to find an optimal shape of a system (or an object) under design. From mathematical point of view a shape is an open subset (for brevity – domain) in $\mathbf{R}^n$. It is required to maximize a cost functional over a class of admissible domains. The main feature of the problem is that a functional to be maximized depends on a domain by means of a solution of a boundary value problem.

In section 1 of this paper the well-known shape optimization problem is formulated and definition of their generalized solutions is given. In sections 2, 3 two auxiliary optimization problems are considered and its connection with the problem of section 1 is established. In section 4 an algorithm of approximate shape optimization is described. In this algorithm the problem of determination of an optimal shape is replaced by a family of coefficient optimization problems for an elliptic equation.

2. A shape optimization problem and its generalized solutions

We will use the following notations:

- D is a fixed bounded open subset of the space $\mathbf{R}^n$;
- ∂D is the boundary of the set D;
- meas(Ω) is the Lebesgue measure of a set Ω;
- S is a fixed positive number, $S \leq \text{meas}(D)$; $\overset{\circ}{\mathrm{W}}\,{}^1_2(\Omega)$ is the Sobolev space of functions which are equal to zero on $\partial\Omega$, Ω is an open set in $\mathbf{R}^n$, Adams (1975); we will assume that the space $\overset{\circ}{\mathrm{W}}\,{}^1_2(\Omega)$ is embedded in $\mathrm{W}^1_2(\mathbf{R}^n)$ the following way: all functions from $\overset{\circ}{\mathrm{W}}\,{}^1_2(\Omega)$ are equal to zero on $\mathbf{R}^n \setminus \Omega$;
- ∇u is the gradient of a function u;
- supp(u) is the support of a function u;
- $||\mathrm{u}|\mathrm{X}||$ is a norm of a function u in a space X;
- $\triangle$ is the Laplace operator.

We will consider the problem of determination of the cross-section of a prismatic bar having maximum torsional rigidity for Saint-Venant model of torsion, Washizu (1982). Let Ω be an open simple-connected set in $\mathbf{R}^2$, corresponding to the cross-section of a prismatic bar. The state function of our system is the Prandtl function - the solution of the following boundary value problem:

$$-\triangle u = 1 \quad \text{in} \quad \Omega, \quad u\,|_{\partial\Omega} = 0 \tag{1}$$

We denote the generalized solution of the boundary value problem (1) as $u(\Omega)$. Let us remind, Rektorys (1980), that the generalized solution of the problem (1) is the function from the space $\overset{\circ}{\mathrm{W}}\,{}^1_2(\Omega)$ satisfying the variational identity

$$\forall v \in \overset{\circ}{\mathrm{W}}\,{}^1_2(\Omega) \int_\Omega \nabla u \cdot \nabla v \ \mathrm{dx} = \int_\Omega v \ \mathrm{dx}. \tag{2}$$

or, equivalent, minimizing the functional $E_1(u) = \int_D \{(\nabla u)^2 - 2 \cdot u\}\ \mathrm{dx}$ over the space $\overset{\circ}{\mathrm{W}}\,{}^1_2(\Omega)$. So the state of the system depends on a domain Ω. As a cost functional of a domain we accept the value which is proportional to the torsional rigidity of the bar with cross-section Ω:

$$R_1(\Omega) = \int_D u(\Omega)\ \mathrm{dx}.$$

We define the class of admissible sets as

$$\Phi = \{\Omega : \Omega \subseteq D, \quad \text{meas}(\Omega) \leq S\}.$$

PROBLEM P1. It is required to determine an admissible set, maximizing the functional R_1 over the class of sets Φ.

The question of existence of a solution of the problem P1 is nontrivial, Pironneau (1984), Osipov, Suetov (1984), so that it is convenient to accept the following definition.

DEFINITION. We call a generalized solution of the shape optimization problem a weak limit point of a sequence of boundary value problems solutions $\{u(\Omega_i)\}$ in the space $\overset{\circ}{W}{}^1_2(D)$, where $\{\Omega_i\}$ is a maximizing sequence for the problem P1.

STATEMENT 1 *There exists a generalized solution of the problem P1.*

Proof. A sequence $\{u(\Omega_i)\}$ $\Omega_i \subseteq D$, is bounded in space $\overset{\circ}{W}{}^1_2(D)$, Rektorys (1980), it has a weak limit point ∎

REMARK 1 *It is possible to write the torsional rigidity functional in the following forms:*

$$R_1(\Omega) = \int_\Omega u(\Omega)\, dx = \int_\Omega (\nabla u(\Omega))^2\, dx = -E_1(u(\Omega)),$$

using the variational identity (2) for $v = u = u(\Omega)$

3. A minimum energy problem

Let us formulate an auxiliary problem of optimization (see Pironneau 1984, ch. 3.2.2).

PROBLEM P2. Let $U = \{u \in \overset{\circ}{W}{}^1_2(D) : \text{meas}(\text{supp}(v)) \leq S\}$ and $F(u) = -E_1(u)$. It is required to determine a function $u^* \in U$ such that $F(u^*) = \max_{u \in U} F(u)$ (so the energy functional $E_1(u^*)$ has a minimum value).

STATEMENT 2 *There exists a solution of the problem P2. A weak limit point of a maximizing sequence of the problem P2 is a strong limit point of the sequence.*

Proof. Existence follows from coerciveness and weakly lower semicontinuity of the functional $E_1(u)$ on $\overset{\circ}{W}{}^1_2(D)$, from compactness of the imbedding $\overset{\circ}{W}{}^1_2(D)$ into $L_2(D)$ and from strong closeness of the set U in $L_2(D)$ (see Pironneau (1984), ch. 3.2.2). Let us prove the strong convergence. Let $\{u_i\}$ be a sequence of functions from the set U, maximizing the functional $-E_1(u)$ over U and $u_i \rightharpoonup u^*$ (as $i \to \infty$) weakly in $\overset{\circ}{W}{}^1_2(D)$. Then $E_1(u_i) \longrightarrow E_1(u^*)$ (as $i \to \infty$), therefore $\|\nabla u_i | L_2(D)\| \longrightarrow \|\nabla u^* | L_2(D)\|$ (as $i \to \infty$). This implies strong convergence of the maximizing sequence in $\overset{\circ}{W}{}^1_2(D)$ ∎

REMARK 2 *By the definition of the class* Φ *from* $\Omega \in \Phi$ *it follows that* $u(\Omega) \in U$, *and by remark 1 we have* $R_1(\Omega) = -E_1(u(\Omega))$, *therefore*

$$\sup_{\Omega \in \Phi} R_1(\Omega) \leq \max_{u \in U}(-E_1(u)).$$

LEMMA 1 *Let* $w \in \mathrm{W}_2^1(\mathbf{R}^{\mathrm{n}})$ *and* $\mu(\delta; w) = \mathrm{meas}\{x \in \mathbf{R}^{\mathrm{n}} : w(x) > \delta\}$ *for* $\delta > 0$. *Then the following estimations hold:*
(a) for small $\tau > 0$ *we have*

$$\mu(\delta + \tau; w) \leq \mu(\delta; w) - \frac{\tau^2 C_n \cdot (\mu(\delta; w))^{\frac{2n-2}{n}}}{\|w|\mathrm{W}_2^1(\mathbf{R}^{\mathrm{n}})\|^2}$$

(b) $\mathrm{meas}(\mathrm{supp}(w)) \geq \tilde{C}_n \left(\frac{\|w|\mathrm{L}_2(\mathbf{R}^{\mathrm{n}})\|}{\|w|\mathrm{W}_2^1(\mathbf{R}^{\mathrm{n}})\|} \right)^{\frac{n}{n-1}}$,

and if $\mathrm{meas}(\mathrm{supp}(w)) < \infty$ *then for small positive number* δ *we have*

$$\mu(\delta, w) \geq \tilde{C}_n \left(\frac{\|w|\mathrm{L}_2(\mathbf{R}^{\mathrm{n}})\|^2 - \delta^2 \mathrm{meas}(\mathrm{supp}(w))}{\|w|\mathrm{W}_2^1(\mathbf{R}^{\mathrm{n}})\|^2} \right)^{\frac{n}{2n-2}}$$

where coefficients $C_n, \tilde{C}_n > 0$ *depend on the dimensionality of the space* $\mathbf{R}^{\mathrm{n}}$ *only.*

Proof. Without loss of generality we can assume that the function w is nonnegative. Let function $\tilde{w}$ be a spherical-symmetric rearrangement of the function w, Polya (1948). It is known, Polya (1948), that $\|w|\mathrm{W}_2^1(\mathbf{R}^{\mathrm{n}})\| \leq \|w|\mathrm{W}_2^1(\mathbf{R}^{\mathrm{n}})\|$ and by definition $\mu(\delta; \tilde{w}) = \mu(\delta; w)$, therefore it is sufficient to prove the estimations for spherical-symmetric function $\tilde{w}$. This function is uniquely defined by function $\mu(\delta; w)$. We omit details. ■

LEMMA 2 *Let* $u^* \in \overset{\circ}{\mathrm{W}}{}_2^1(D)$ *and* $\mathrm{meas}(\mathrm{supp}(u^*)) \leq S$. *Then there exists a family of domains* $\{\Omega_\epsilon\}$, $\Omega_\epsilon \in \Phi$ *and a family of functions* $\{u_\epsilon\}$, $u_\epsilon \in \overset{\circ}{\mathrm{W}}{}_2^1(\Omega_\epsilon)$, $u_\epsilon \longrightarrow u^*$ *(as* $\epsilon \to 0$*) strongly in* $\overset{\circ}{\mathrm{W}}{}_2^1(D)$.

Proof. Let $\{\tilde{u}_\epsilon\}$ be a family of functions from the space $\dot{C}^\infty(D)$, approximating u^* in $\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})$: $\|u^* \text{-} \tilde{u}_\epsilon|\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})\| \leq \epsilon$. Then $\|u^* \text{-} \tilde{u}_\epsilon|\mathrm{L}_2(\mathrm{D})\| \leq \epsilon$. Since $\mathrm{meas}(\mathrm{supp}(u^*)) \leq S$, the inequality

$$\mu(\delta, \tilde{u}_\epsilon) \leq S + \frac{\|u^* \text{-} \tilde{u}_\epsilon|\mathrm{L}_2(\mathrm{D})\|^2}{\delta^2} \leq S + \frac{\epsilon^2}{\delta^2}$$

holds. By inequality (a) from lemma 1 for $\delta > 0$, $\tau = \delta$ we have

$$\mu(2\delta, \tilde{u}_\epsilon) \leq \mu(\delta, \tilde{u}_\epsilon) - \delta^2 C(\delta, \tilde{u}_\epsilon) \leq S + \frac{\epsilon^2}{\delta^2} - \delta^2 C(\delta, \tilde{u}_\epsilon),$$

where $\quad C(\delta,\tilde{u}_\epsilon)=\dfrac{C_n\cdot(\mu(\delta;\tilde{u}_\epsilon))^{\frac{2n-2}{n}}}{\|\tilde{u}_\epsilon|\mathrm{W}_2^1(\mathbf{R}^n)\|^2}$

It is evident that $\mathrm{meas}(\mathrm{supp}(\tilde{u}_\epsilon))\le \mathrm{meas}(D)$ and by inequality (b) from lemma 1 we have

$$C(\delta,\tilde{u}_\epsilon)\ge\frac{C_n\cdot\tilde{C}_n\cdot(\|\tilde{u}_\epsilon|\mathrm{L}_2(\mathbf{R}^n)\|-\delta^2\mathrm{meas}(D))}{\|\tilde{u}_\epsilon|\mathrm{W}_2^1(\mathbf{R}^n)\|^4}$$

Using the triangle inequality we obtain $\|\tilde{u}_\epsilon|\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})\|\le 2\cdot\|u^*|\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})\|$ and $\|\tilde{u}_\epsilon|\mathrm{L}_2(\mathrm{D})\|\ge\frac{1}{2}\|u^*|\mathrm{L}_2(\mathrm{D})\|$ for $\epsilon\le\frac{1}{2}\|u^*|\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})\|$, so that for such ϵ the following inequality holds

$$C(\delta,\tilde{u}_\epsilon)\ge\frac{C_n\cdot\tilde{C}_n\cdot(\frac{1}{4}\|u^*|\mathrm{L}_2(\mathbf{R}^n)\|^2-\delta^2\mathrm{meas}(D))^2}{16\cdot\|u^*|\mathrm{W}_2^1(\mathbf{R}^n)\|^4}$$

It is possible to assume that $C(\delta,\tilde{u}_\epsilon)\ge C^*>0$ for $\delta^2\le\frac{1}{4}\mathrm{meas}(D)\|u^*|\mathrm{L}_2(\mathbf{R}^n)\|^2$ and $\epsilon\le\frac{1}{2}\|u^*|\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})\|$, where a number C^* does not depend on δ, ϵ. In virtue of (3.) for sufficiently small positive ϵ and $\delta(\epsilon)=\left(\frac{\epsilon^2}{C^*}\right)^{\frac{1}{4}}$ the inequality $\mu(\delta(\epsilon),\tilde{u}_\epsilon)\le S$ is satisfied. We define a family of functions

$$u_\epsilon(x)=\begin{cases}\tilde{u}_\epsilon(x)-\delta(\epsilon) & \text{if}\quad \tilde{u}_\epsilon(x)>\delta(\epsilon),\\ 0 & \text{otherwise.}\end{cases}$$

and corresponding family of domains $\Omega_\epsilon=\{x:u_\epsilon(x)>\delta(\epsilon)\}$. By continuity of the function u_ϵ the set Ω_ϵ is open and $u_\epsilon\in\overset{\circ}{\mathrm{W}}{}_2^1(\Omega_\epsilon)$. It is clear that $\Omega_\epsilon\in\Phi$ and

$\|\tilde{u}_\epsilon|\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})\|\le\|u_\epsilon|\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})\|,\ \|\tilde{u}_\epsilon\text{-}u_\epsilon|\mathrm{L}_2(\mathrm{D})\|\le\delta(\epsilon)(\mathrm{meas}(D))^{\frac{1}{2}}$.

Thus $u_\epsilon\longrightarrow u^*$(as $\epsilon\to 0$) strongly in $\mathrm{L}_2(\mathrm{D})$, $\{u_\epsilon\}$ is bounded in $\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})$ and therefore $u_\epsilon\rightharpoonup u^*$(as $\epsilon\to 0$) weakly in $\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})$. Since $\limsup_{\epsilon\to 0}\|u_\epsilon|\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})\|=\lim_{\epsilon\to 0}\|\tilde{u}_\epsilon|\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})\|$ hencs $u_\epsilon\longrightarrow u^*$(as $\epsilon\to 0$) strongly in $\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})$ follows from the weak convergence. ■

THEOREM 1 *The set of generalized solutions of the problem P1 is equal to the set of solutions of the problem P2.*

Proof. (a) A solution of problem P2 is a generalized solution of problem P1. Let u^* be a solution of problem P2. Let $\{\Omega_\epsilon\}$ be a family of domains, $\Omega_\epsilon\in\Phi$, and $\{u_\epsilon\}$ be a family of functions, $u_\epsilon\in\overset{\circ}{\mathrm{W}}{}_2^1(\Omega_\epsilon)$, $u_\epsilon\longrightarrow u^*$(as $\epsilon\to 0$) strongly in $\overset{\circ}{\mathrm{W}}{}_2^1(\mathrm{D})$. Existence of the families follows from lemma 2. It is obvious that

$\lim_{\epsilon\to 0}$
$E_1(u_\epsilon) = E_1(u^*)$. In virtue of remark 1 and property of solutions of (2) we have $R_1(\Omega_\epsilon) = -E_1(u(\Omega_\epsilon)) \geq -E_1(u_\epsilon)$, therefore $\lim_{\epsilon\to 0} R_1(\Omega_\epsilon) \geq -E_1(u^*)$. But by remark 2 $R_1(\Omega_\epsilon) \leq -E_1(u^*)$, therefore

$$\lim_{\epsilon\to 0} R_1(\Omega_\epsilon) = \lim_{\epsilon\to 0} -E_1(u(\Omega_\epsilon)) = \lim_{\epsilon\to 0} -E_1(u_\epsilon) = \tag{3}$$

$$= \sup_{\Omega\in\Phi} R_1(\Omega) = \max_{u\in U}(-E_1(u)).$$

The variational identity (2) implies equality

$$E_1(u(\Omega_\epsilon)) - E_1(u_\epsilon) = \|\nabla \mathrm{u}(\Omega_\epsilon)\text{-}\nabla u_\epsilon|\mathrm{L}_2(\mathrm{D})\ \|^2.$$

Since the seminorm $(\int_D (\nabla u(x))^2\ \mathrm{dx})^{\frac{1}{2}}$ is equivalent to the standard norm in $\overset{\circ}{\mathrm{W}}\,{}^1_2(\mathrm{D})$, so $u(\Omega_\epsilon)\longrightarrow u^*$(as $\epsilon\to 0$) strongly in $\overset{\circ}{\mathrm{W}}\,{}^1_2(\mathrm{D})$ and u^* is a generalized solution of the problem P1.
(b) A generalized solution of problem P1 is a solution of problem P2. Let $\{\Omega_i\}$ be a maximizing sequence for the problem P1 and $u(\Omega_i)\rightarrow u^\#$(as $i\to\infty$) weakly in $\overset{\circ}{\mathrm{W}}\,{}^1_2(\mathrm{D})$. Since $\Omega_i \in \Phi$ it follows that $u(\Omega_i) \in U$ and $u(\Omega_i)\longrightarrow u^\#$(as $i\to\infty$) strongly in $\mathrm{L}_2(\mathrm{D})$, hence $u^\# \in U$. By (3) we have

$$\lim_{\epsilon\to 0} R_1(\Omega_i) = \lim_{\epsilon\to 0} -E_1(u(\Omega_i)) = \max_{u\in U}(-E_1(u)) \geq -E_1(u^\#).$$

From the weak lower semicontinuity of the functional E_1 it follows that $\lim_{\epsilon\to 0}$ $-E_1(u(\Omega_i)) \leq -E_1(u^\#)$, therefore $u^\#$ is a solution of the problem P2. ∎

COROLLARY 1 *Weak limit points of maximizing sequences of the problem P1 are strong limit points of the sequences (hence generalized solutions of the problem P1 always are strong limit points of corresponding maximizing sequences).*

Proof. Using notations of theorem 1 proof (b) we have $\lim_{i\to\infty} -E_1(u(\Omega_i)) = -E_1(u^\#)$, and it follows that $\|\nabla u_i|\mathrm{L}_2(\mathrm{D})\ \|\longrightarrow\|\nabla u^\#|\mathrm{L}_2(\mathrm{D})\ \|$ (as $i\to\infty$). The strong convergence follows from the weak convergence and from the convergence of the norms. ∎

4. A coefficient optimization problem

Let A be a positive number. We define a set of functions

$$K_A = \{k \in L_\infty(D) : \textstyle\int_D (A - k(x))\ \mathrm{dx} \geq A\cdot S,$$
$$\forall x \in D,\quad 0 \leq k(x) \leq A\}$$

We call K_A the set of admissible coefficients. It is obvious that the set K_A is convex and it is weakly compact in space L_p for $1 \leq p < \infty$. For every admissible coefficient $k(x)$ we have a boundary value problem

$$-\triangle u + k\cdot u = 1 \ \text{ in }\ D,\ \ u\,|_{\partial D}= 0. \tag{4}$$

We will denote as $u(k)$ the generalized solution of the problem (4). Let us remind that the generalized solution of the boundary value problem (4) minimizes the functional

$$E_2(u,k) = \int_D \{(\nabla u)^2 + k{\cdot}u^2 - 2{\cdot}u\}\ \mathrm{dx}$$

for the argument u over the space $\overset{\circ}{\mathrm{W}}\,{}^1_2(\mathrm{D})$, Rektorys (1980), and satisfies the variational identity

$$\forall v \in \overset{\circ}{\mathrm{W}}\,{}^1_2(\mathrm{D}) \quad \int_D \{\nabla u{\cdot}\nabla v + k{\cdot}u{\cdot}v\}\ \mathrm{dx} = \int_D v\ \mathrm{dx}. \tag{5}$$

We define a functional on the set of admissible coefficients:

$$R_2(k) = \int_D u(k)\ \mathrm{dx}.$$

REMARK 3 *We note that equalities*

$$-E_2(u(k),k) = \int_D \{(\nabla u(k))^2 + k{\cdot}u^2(k)\}\ \mathrm{dx} =$$

$$= \int_D u(k)\ \mathrm{dx} = R_2(k)$$

follow from the variational identity (5) for $v = u(k)$

PROBLEM P3$_{\mathrm{A}}$. It is required to determine an admissible coefficient maximizing the functional R_2.

STATEMENT 3 *Problem P3$_{\mathrm{A}}$ has a solution.*

The statement 3 follows from weak compactness of K_A and from weak continuity of the mapping $k \mapsto u(k)$. ■

THEOREM 2 *Let k_A be a solution of the problem P3$_{\mathrm{A}}$, $u_A = u(k_A)$. Then there exists number h_A such that*

$$k_A(x) = \begin{cases} 0 & \text{if } \ u_A(x) > h_A, \\ A & \text{otherwise.} \end{cases}$$

The number h_A is uniquely defined from the condition: $\mathrm{meas}\{x \in D : u_A(x) > h_A\} = S$. *In addition, $h_A \longrightarrow 0$ (as $A \to \infty$).*

The proof follows from the remark: in virtue of (3) the problem P3$_{\mathrm{A}}$ is equivalent to the minimization problem for the functional $E_2(u,k)$ over both arguments.■

LEMMA 3 *Let k_A be a solution of the problem P3$_{\mathrm{A}}$ and $u_A = u(k_A)$ for each positive number A. Thhe measure or support of each limit function of the family $\{u(A)\}$ (as$A \to 0$) is less or equal to the number S.* ■

THEOREM 3 *Let k_A be a solution of the problem $P3_A$ and $u_A = u(k_A)$ for each positive number A. Then a limit point u^* of the family $\{u_A\}$ (as $A \to \infty$) is a solution of the problem P2.*

Proof. By lemma 3 we have $u^* \in U$, therefore it is sufficient to prove that $E_1(u^*) = \inf_{u \in U} E_1(u)$. Let $v \in U$ and χ_v be the characteristic function of the set $\mathbf{R}^n \setminus \mathrm{supp}(v)$. Then the function $A \cdot \chi_v$ is an admissible control for the problem $P3_A$ and the following inequalities hold:

$$\forall v \in U \; E_1(u_A) \leq E_2(u_A, k_A) \leq E_2(v, A \cdot \chi_v) = E_1(v).$$

By weak lower semicontinuity of the functional $E_1(u)$ we have inequalities

$$E_1(u^*) \leq \liminf_{A \to \infty} E_2(u(A), A \cdot \chi_v) \leq E_1(v).$$

The theorem is proved. ■

5. An algorithm of shape optimization

Theorems 1, 2 and 3 give us a possibility to propose the following method for calculation of a suboptimal shape for our problem:
1) choosing a large number A;
2) solving the problem $P3_A$;
3) taking a set $\Omega = \{x \in D : u^*(x) > h\}$ as a suboptimal shape (according to notations of the theorem 3).

It is obvious that problem $P3_A$ is non-convex. We describe an algorithm for computation of a function, satisfying to the necessary condition of optimality for the problem $P3_A$ (theorem 2). Let $\chi(u, \delta)$ be the characteristic function of the level set $\{x \in D : u(x) \leq \delta\}$ for $\delta > 0$ and $u \in \overset{\circ}{W}{}^1_2(D)$, so that

$$\chi(u, \delta; x) = \begin{cases} 0 & \text{if } \; u(x) > \delta, \\ 1 & \text{otherwise.} \end{cases}$$

Let $u_A(\delta)$ be the solution of (4) for $k = A \cdot \chi(u_A(\delta), \delta)$ (hence $u_A(\delta) = u(A \cdot \chi(u_A(\delta), \delta)))$.

THEOREM 4 *Let $u_0 = u(D)$, $u_{i+1} = u(A \cdot \chi(u_i, \delta))$. Then the sequence $\{u_i\}$ is pointwise monotone converging to $u_A(\delta)$, the function $u_A(\delta)$ is continuous depending on δ in $\overset{\circ}{W}{}^1_2(D)$, the function $\chi(u_A(\delta), \delta)$ is continuous depending on δ in the space $L_1(D)$.*

Proof of the theorem 4 is based on pointwise monotonic decreasing of the sequence $\{u_i\}$. ■

It is sufficient to find δ_A from the equation

$$\mathrm{meas}\{x \in D : u_A(\delta) > \delta\} = S, \tag{6}$$

for determination of a couple of functions $(k_A, u(k_A))$, satisfying the condition of the theorem 2. since $k_A = \chi(u_A(\delta_A), \delta_A)$, $u(k_A) = u_A(\delta_A)$. In this case we have to solve the boundary value problem of the type (4) repeatedly.

In the conclusion we formulate a convergence theorem for discrete approximations.

THEOREM 5 *Let $\{W_m\}$ be a sequence of subspaces in $\overset{\circ}{W}{}^1_2(D)$, $W_m \longrightarrow \overset{\circ}{W}{}^1_2(D)$ (as $m \to \infty$) strongly in $\overset{\circ}{W}{}^1_2(D)$, and $u_m(A)$ be a solution of the analog of the problem $P3_{\mathrm{A}}$ in the space W_m.*

Then limit points of the family $\{u_m(A)\}$ (as $m \to \infty$, $A \to \infty$) are solutions of the problem P2.

6. Conclusion

In this work an algorithm for solving shape optimization problem is proposed and justified. The important feature of the algorithm is that corresponding boundary value problems can be solved on the same domain and on the same grid, with the distinction in the lowest term coefficients of equations only. The idea of the algorithm is applicable to other shape optimization problems for minimization of the energy functional of a system.

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