

**The Upper Boundary Approach
to Constrained Discrete Time Optimal Control**

by

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In the paper the basic geometrical concepts of the upper boundary approach to discrete time optimal control in the presence of mixed state and control constraints are introduced. Relations are discussed between the upper boundary approach and the main optimization principles: the maximum principle, the dynamic programming, and the Krotov conditions.

1. Introduction

Let us consider the following problem:

$$\text{maximize} \quad J = \sum_{i=0}^{N-1} r_i(x_i, u_i), \quad (1a)$$

subject to:

$$x_{i+1} = f_i(x_i, u_i), \quad (1b)$$

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$$(x_i, u_i) \in V_i, \quad i = 0, 1, \dots, N-1, \quad (1c)$$

$$x_N \in V_N, \quad (1d)$$

where $u_i \in R^m$ are controls or decisions, $x_i \in R^n$ are states, $V_i \subseteq R^{n+m}$, $V_N \subseteq R^n$ are given constraint sets, $r_i: W_i \rightarrow R^1$, $f_i: W_i \rightarrow R^n$, $W_i \subseteq R^{n+m}$.

Different approaches are possible to solve the above problem. It can be treated as a mathematical programming problem. This approach is quite general in the sense that minimal requirements for the problem functions are needed. Its main drawback is the necessity to cope with the usually big dimension of the problem caused by the multiplicative effect of the number of stages N . This usually requires application of different decomposition techniques.

Development of the control theory showed that actually the set containing optimal solutions may be considerably restricted. This was expressed in the famous Bellman principle of optimality [1] which asserts that only those solutions which form a so called return function (optimal value function, Bellman function) can contain the optimal solutions. Also in the forward direction the similar assertion is true. This is often called the Halkin principle of optimal evolution [16]. Halkin's paper [16] made it also evident that the Pontryagin maximum principle [29, 30] is a way of working with a forward optimal value function using a penalty technique. The sets of values of those backward and forward optimal value functions are upper boundaries of the sets of extended states which will be defined in Sec. 2.

Thus the upper boundaries of the sets of extended states form an important subset which is actually dealt with in main control theoretical optimization techniques. Yet apart from dynamic programming [1, 2] where the values of the return functions are directly calculated and where some their properties like differentiability are explored [19, 48] there has been relatively little interest in a characterization of the upper boundaries in the optimal control literature. Opposite to this, in the related area of mathematical programming, the upper boundaries — there called the perturbation functions or marginal functions — have been the subject of intensive studies over the last ten years see e.g. [15, 16, 39, 40, 41].

It is perhaps worth noting that in mathematical programming even smaller sets containing optimal solutions are characterized. These are the sets of critical points connected with the kernel of the Lagrangean gradient [21, 22]. So far there is a lack of practical exploitation of these interesting results.

We believe that examination of the upper boundary properties may contribute to better understanding and construction of efficient algorithm for solving the problem (1). The sketch of perspectives for the unconstrained case was presented in an earlier book [27] where not only a number of known results were shown to be easily derived as a consequence of intro-

duction of the upper boundaries but also a new generalized maximum principle was proved.

In this paper we survey the constrained case. Generalization of some results from [27] is given. Connections with other main methods are discussed.

The organization of the paper is as follows. Sec. 2 contains preliminaries and basic geometrical concepts connected with the problem (1). This set of notions form the basis of our upper boundary approach. In Sec. 3 connections between the upper boundary approach and different versions of the maximum principle are discussed, while in Sec. 4 they are related to dynamic programming. In Sec. 5 we show that the Krotov functions introduced in his necessary and sufficient conditions [23] are actually supports to the upper boundaries. The paper is concluded in Sec. 6. A review of conditions which lead to different properties of the upper boundary functions is given in the Appendix which constitutes Sec. 7.

Throughout the paper we denote by subscript the time and by superscript the element of the vector, e.g. b_i^j is the j th element of the vector b at time i .

2. Basic geometrical concepts

With the set V_i of admissible control and states at the stage i we relate two sets, see Fig. 1. One is the set of admissible controls at the stage i for a given x_i defined by:

$$U_i(x_i) = \{u_i : (x_i, u_i) \in V_i\} = S_{x_i}(V_i), \quad (2)$$

where $S_{x_i}(V_i)$ is the projection of the intersection of the set V_i with the plane $x_i = \text{constant}$ on the subspace of u_i . The other is the set of admissible states at stage i defined by:

$$X_i = \{x_i : \exists u_i | (x_i, u_i) \in V_i\} = P_{x_i}(V_i), \quad (3)$$

where $P_{x_i}(V_i)$ is the projection of the set V_i on the subspace of x_i .

Varying u_i in $U_i(x_i)$ a set of states reachable from a given $x_i \in X_i$ can be obtained:

$$Y_{i+1}(x_i) = \bigcup_{u_i \in U_i(x_i)} f_i(x_i, u_i) = f_i(x_i, U_i(x_i)), \quad (4)$$

and varying additionally x_i in $X_i \cap Y_i$ a set of the reachable states at stage $i+1$ is recursively defined by:

$$\begin{aligned} Y_{i+1} &= \bigcup_{x_i \in X_i \cap Y_i} Y_{i+1}(x_i) = \{f_i(x_i, u_i) \in V_i \cap (Y_i \times R^m)\} = \\ &= f_i(X_i \cap Y_i, \bigcup_{x_i \in X_i \cap Y_i} U_i(x_i)), \quad i = 0, 1, \dots, N-1, \end{aligned} \quad (5a)$$

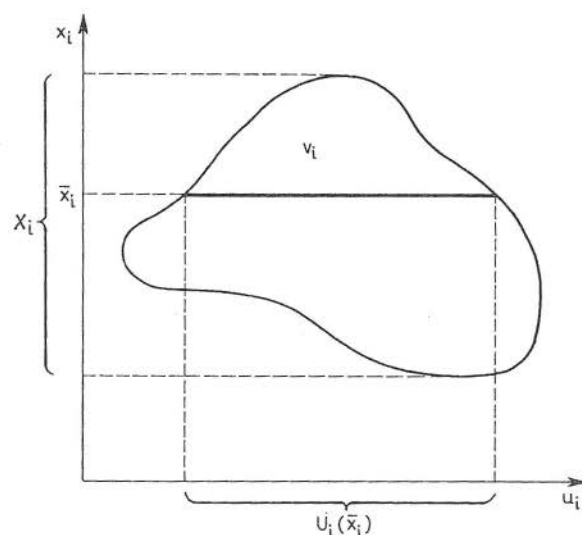


Fig. 1. The set V_i of admissible controls, the set X_i of admissible states, and the set $U_i(x_i)$ of admissible controls for a given x_i .

$$Y_0 = X_0, \quad (5b)$$

Let us define the extended state:

$$\hat{x}_i = \begin{bmatrix} x_i^0 \\ x_i \end{bmatrix}, \quad (6)$$

where x_i^0 is defined recursively:

$$x_0^0 = 0, \quad (7a)$$

$$x_{i+1}^0 = x_i^0 + r_i(x_i, u_i) \quad i = 0, 1, \dots, N-1. \quad (7b)$$

The values u_i , x_i , and x_{i+1} satisfy (1b-c).

Now, similar as before we can define the set of extended states at the stage $i+1$ reachable from a given extended state $\hat{x}_i = [x_i^0, x_i]^T$:

$$\hat{Y}_{i+1}(\hat{x}_i) = [x_i^0 + r_i(x_i, U_i(x_i))] \times f_i(x_i, U_i(x_i)), \quad (8)$$

and the set of reachable extended states at stage $i+1$:

$$\hat{Y}_{i+1} = \bigcup_{\hat{x}_i \in (R \times X_i) \cap \hat{Y}_i} \hat{Y}_{i+1}(\hat{x}_i), \quad (9a)$$

$$\hat{Y}_0 = \{0\} \times X_0. \quad (9b)$$

We define on the set $Y_{i+1}(x_i)$ the function $ub_{i+1}(\hat{x}_i, \cdot): R^n \rightarrow R^1$ as:

$$ub_{i+1}(\hat{x}_i, x_{i+1}) = \sup r_i(x_i, u_i) + x_i^0, \quad (10a)$$

over all u_i satisfying:

$$\begin{aligned} u_i &\in U_i(x_i), \\ f_i(x_i, u_i) &= x_{i+1}, \end{aligned} \quad (10b)$$

and call it the smaller boundary function. On the set Y_{i+1} we define the function $UB_{i+1}: Y_{i+1} \rightarrow R^1$, called the greater upper boundary function, as:

$$UB_{i+1}(x_{i+1}) = \sup \sum_{j=0}^i r_j(x_j, u_j), \quad (11a)$$

over all (x_j, u_j) , $j = 0, 1, \dots, i$ satisfying:

$$\begin{aligned} (x_j, u_j) &\in V_j, \\ f_j(x_j, u_j) &= x_{j+1} \quad j = 0, 1, \dots, i, \end{aligned} \quad (11b)$$

or equivalently:

$$UB_{i+1}(x_{i+1}) = \sup [r_i(x_i, u_i) + UB_i(x_i)], \quad (12a)$$

over all (x_i, u_i) satisfying:

$$\begin{aligned} (x_i, u_i) &\in V_i, \\ f_i(x_i, u_i) &= x_{i+1}, \\ x_i &\in Y_i, \end{aligned} \quad (12b)$$

for $i = 0, 1, \dots, N-1$, and additionally:

$$UB_0(x_0) = 0. \quad (12c)$$

The value $ub_{i+1}(\hat{x}_i, x_{i+1})$ is the upper boundary of all values $r_i(x_i, u_i) + x_i^0$ over all $u_i \in U_i(x_i)$ satisfying $f_i(x_i, u_i) = x_{i+1}$. Therefore in the space of extended states \hat{x}_{i+1} the point $(ub_{i+1}(\hat{x}_i, x_{i+1}), x_{i+1}^f)^T$ belongs to the upper boundary—in the direction of x_{i+1}^0 —of the set $\hat{Y}_{i+1}(\hat{x}_i)$ of extended states reachable from \hat{x}_i , see Fig. 2.

The value $UB_{i+1}(x_{i+1})$ is the upper boundary of the values of the partial criterion $\sum_{j=0}^i r_j(x_j, u_j)$ over all admissible (x_i, u_j) , $j = 0, 1, \dots, i$ which satisfy the equation of motion $f_j(x_j, u_j) = x_{j+1}$. Therefore in the space of extended states \hat{x}_{i+1} the point $(UB_{i+1}(x_{i+1}), x_{i+1}^f)^T$ belongs to the upper boundary—in the direction of x_{i+1}^0 —of the set \hat{Y}_{i+1} of reachable extended states at stage $i+1$, see Fig. 2.

It is easy to see that the optimal extended states are situated at both upper boundaries at every stage. Otherwise it could be possible to find u_i in the former case and a sequence of (x_j, u_j) in the latter which would lead to greater values of x_{i+1}^0 . This is actually the Halkin principle of optimal evolution [17]. Particularly we have:

$$\sup J = \sup_{x_i \in Y_i} UB_i(x_i).$$

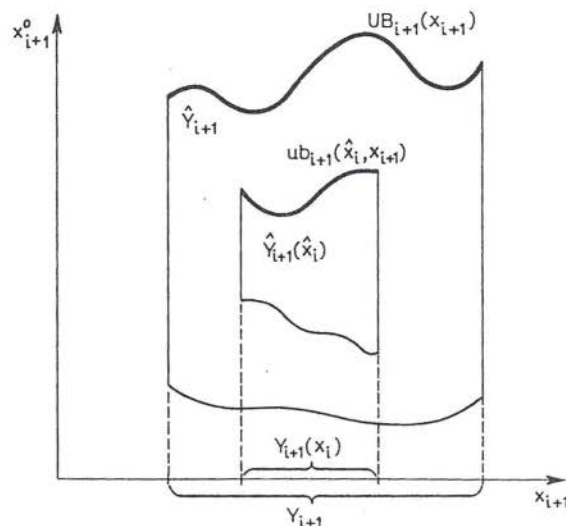


Fig. 2. The set $Y_{i+1}(x_i)$ of states reachable from a given $x_i \in X_i$, the set Y_{i+1} of reachable states, the smaller upper boundary functions $ub_{i+1}(\hat{x}_i, x_{i+1})$, and the upper greater upper boundary function $UB_{i+1}(x_{i+1})$.

A similar construction can be done in the backward direction starting from stage N . For a given $x_{i+1} \in Y_{i+1}$ we introduce the set of states at stage i from which the state x_{i+1} can be reached:

$$RY_i(x_{i+1}) = \{x_i : \exists u_i | (x_i, u_i) \in V_i \cap f_{i+1}^{-1}(x_{i+1})\} = P_x [V_i \cap f_{i+1}^{-1}(x_{i+1})] \quad (13)$$

and the set of states at stage i from which a final state $x_N \in X_N$ can be reached:

$$RY_i = \bigcup_{x_{i+1} \in RY_{i+1}} RY_i(x_{i+1}) = P_x [V_i \cap f_{i+1}^{-1}(RY_{i+1})], \quad (14a)$$

$$i = N-1, N-2, \dots, 0,$$

$$RY_N = V_N. \quad (14b)$$

Let us also introduce the extended state:

$$\tilde{x}_i = \begin{bmatrix} x_i^\square \\ x_i \end{bmatrix}, \quad (15)$$

where:

$$x_N^\square = 0, \quad (16a)$$

$$x_i^\square = x_{i+1}^\square + r_i(x_i, u_i). \quad (16b)$$

The values u_i , x_i , and x_{i+1} satisfy (1b-c).

Now, the set of extended states at stage i from which the state x_{i+1} can be reached is defined as:

$$R\tilde{Y}_i(\tilde{x}_{i+1}) = \bigcup_{x_i \in RY_i(x_{i+1})} [x_{i+1}^\square + r_i(x_i, U_i(x_i))] \times RY_i(x_{i+1}), \quad (17)$$

and the set of extended states at stage i from which a final state $x_v \in X_v$ can be reached as:

$$R\tilde{Y}_i = \bigcup_{\tilde{x}_{i+1} \in R\tilde{Y}_{i+1}} R\tilde{Y}_i(\tilde{x}_{i+1}) \quad i = N-1, N-2, \dots, 0, \quad (18a)$$

$$RY_v = \{0\} \times V_v. \quad (18b)$$

On the set $RY_i(x_{i+1})$ we define the reverse smaller upper boundary $\text{rub}_i(\tilde{x}_{i+1}, \cdot): R^n \rightarrow R^1$ as:

$$\text{rub}_i(x_{i+1}, x_i) = \sup r_i(x_i, u_i) + x_{i+1}^\square, \quad (19a)$$

over all u_i satisfying:

$$u_i \in U_i(x_i), \quad (19b)$$

$$f_i(x_i, u_i) = x_{i+1},$$

and on the set RY_i the reverse greater upper boundary $\text{RUB}_i: R^n \rightarrow R^1$ as:

$$\text{RUB}_i(x_i) = \sup \sum_{j=i}^{v-1} r_j(x_j, u_j), \quad (20a)$$

over all $u_i, (u_j, x_j), j = i+1, i+2, \dots, N-1$, and x_v satisfying:

$$(x_j, u_j) \in V_j,$$

$$f_j(x_j, u_j) = x_{j+1} \quad j = i, i+1, \dots, N-1, \quad (20b)$$

$$x_v \in V_v,$$

for $i = N-1, N-2, \dots, 0$, and additionally:

$$\text{RUB}_v(x_v) = 0. \quad (20c)$$

Similar as before the point $(\text{rub}_i(x_{i+1}, x_i), x_i^T)^T$ is situated at the upper boundary of the set $R\tilde{Y}_i(\tilde{x}_{i+1})$ — in the direction of x_i^\square — and $(\text{RUB}_i(x_i, x_i^T)^T$ at the upper boundary of the set $R\tilde{Y}_i$. And similar to (12) we have:

$$\sup J = \sup_{x_0 \in X_0} \text{RUB}_0(x_0), \quad (21)$$

The function $\text{RUB}_i(x_i)$ is the return (or optimal value) function of dynamic programming.

If x_i^* belongs to the optimal solution of the problem (1) then:

$$\text{UB}_i(x_i^*) + \text{RUB}_i(x_i^*) = \sup J, \quad (22)$$

and for any x_i from $Y_i \cap RY_i$:

$$\text{UB}_i(x_i) + \text{RUB}_i(x_i) \leq \sup J. \quad (23)$$

The above geometrical constructions form the basis of our upper boundary approach. It turns out that the properties of upper boundary functions play in this approach an important role. This is because the methods of solving the problem (1) strongly depend on these properties. Yet even simple properties like continuity or differentiability of upper boundaries are not necessarily secured by simple conditions put on the problem functions r_i , f_i and on the structure of the sets V_i .

To simplify the analysis we confine ourselves in the sequel to the following sets of constraints:

$$V_i = \{(x_i, u_i) : g_i(x_i, u_i) \leq 0, h_i(x_i, u_i) = 0\}, \quad (24a)$$

$$i = 0, 1, \dots, N-1,$$

$$V_N = \{x_N : g_N(x_N) \leq 0, h_N(x_N) = 0\}, \quad (24b)$$

where $g_i: W_i \rightarrow R^{k_i}$, $h_i: W_i \rightarrow R^{l_i}$. To simplify the notation we assume that $V_i \subseteq W_i$.

Conditions guaranteeing some properties of the upper boundary functions are given in the Appendix. The proofs will be given in a forthcoming paper.

3. The upper boundary and the maximum principles

Let us assume that the functions r_i , f_i , h_i , g_i , $i = 0, 1, \dots, N-1$, and N — when appropriate, are differentiable and a solution (x_i^*, u_i^*) , $i = 0, 1, \dots, N-1$, x_N^* is finite and satisfies the equation of motion (1b) and the constraints (1c-d). If this solution is optimal and satisfies a constraint qualification (see e.g. the Appendix), then there exist (Lagrange) multipliers μ_i , λ_i , p_{i+1} such that the following Karush-Kuhn-Tucker necessary conditions hold:

$$\begin{aligned} \nabla \left\{ \sum_{i=0}^{N-1} [r_i(x_i^*, u_i^*) - p_{i+1}^T [f_i(x_i^*, u_i^*) - x_{i+1}^*] - \mu_i^T h_i(x_i^*, u_i^*) - \right. \\ \left. - \lambda_i^T g_i(x_i^*, u_i^*) \right\} - \mu_N^T h_N(x_N^*) - \lambda_N^T g_N(x_N^*) = 0, \end{aligned} \quad (25a)$$

$$\lambda_i^T g_i(x_i^*, u_i^*) = 0 \quad \text{for } i = 0, 1, \dots, N-1, \quad (25b)$$

$$\lambda_N^T g_N(x_N^*) = 0,$$

$$\lambda_i \geq 0 \quad \text{for } i = 0, 1, \dots, N. \quad (25c)$$

They can be converted to the following form:

$$\nabla_u H_i(x_i^*, u_i^*, p_{i+1}) - \mu_i^T \nabla_u h_i(x_i^*, u_i^*) - \lambda_i^T \nabla_u g_i(x_i^*, u_i^*) = 0, \quad (26a)$$

$$-p_i^T = \nabla_x H_i(x_i^*, u_i^*, p_{i+1}) - \mu_i^T \nabla_x h_i(x_i^*, u_i^*) - \lambda_i^T \nabla_x g_i(x_i^*, u_i^*), \quad (26b)$$

$$\lambda_i^T g_i(x_i^*, u_i^*) = 0, \quad (26c)$$

$$\lambda_i \geq 0, \quad (26d)$$

for $i = 0, 1, \dots, N-1$, with $p_0 = 0$, and:

$$p_i^T = \mu_i^T \nabla_x h_i(x_i^*) + \lambda_i^T \nabla_x g_i(x_i^*), \quad (26e)$$

$$\lambda_i^T g_i(x_i^*) = 0, \quad (26f)$$

$$\lambda_i \geq 0, \quad (26g)$$

where:

$$H_i(x_i, u_i, p_{i+1}) = r_i(x_i, u_i) - p_{i+1}^T f_i(x_i, u_i), \quad (27)$$

is called in the control literature the hamiltonian.

Since the works of Pontryagin and his associates [3, 4, 5, 30, 31] there exist in the control literature stronger necessary conditions in which that of (26a) is replaced by the stagewise maximization:

$$\sup_{u_i \in U_i(x_i^*)} H_i(x_i^*, u_i, p_{i+1}). \quad (28)$$

The conditions (26) with (26a) replaced by (28) are called the maximum principle. When true, they allow to reduce the number of points suspected for optimality.

However, in the discrete time case the maximum principle does not hold generally, even when assumptions similar to those of the continuous time are taken. To find the appropriate conditions we use our geometrical constructions.

Let us note first that if an optimal solution $(u_0^*, x_0^*, \dots, u_{N-1}^*, x_{N-1}^*, x_N^*)$ exists, then the optimal extended state \hat{x}_{i+1}^* lies on the smaller and upper boundaries, i.e. it is of the form $(\text{ub}_{i+1}(\hat{x}_i^*, x_{i+1}^*), x_{i+1}^{*l})^T$ and $(\text{UB}_{i+1}(x_{i+1}^*), x_{i+1}^{*l})^T$. It may, however, happen that there is no optimal solution even when there are admissible ones. The existence of an optimal solution is closely related to the upper semicontinuity of the upper boundary functions, see Prop. A.2 in the Appendix for sufficient conditions.

Let us then assume that an optimal solution to the problem (1) exists and that there exists a supporting hyperplane to the set $\hat{Y}_{i+1}(\hat{x}_i^*)$ at an optimal extended state \hat{x}_{i+1}^* belonging to the same optimal solution as x_i^* . Let the hyperplane equation be:

$$x_{i+1}^0 = p_{i+1}^T x_{i+1} + c_{i+1}. \quad (29)$$

Then it is a supporting hyperplane if, see also Fig. 3:

$$x_{i+1}^{0*} = p_{i+1}^T x_{i+1}^* + c_{i+1}, \quad (30a)$$

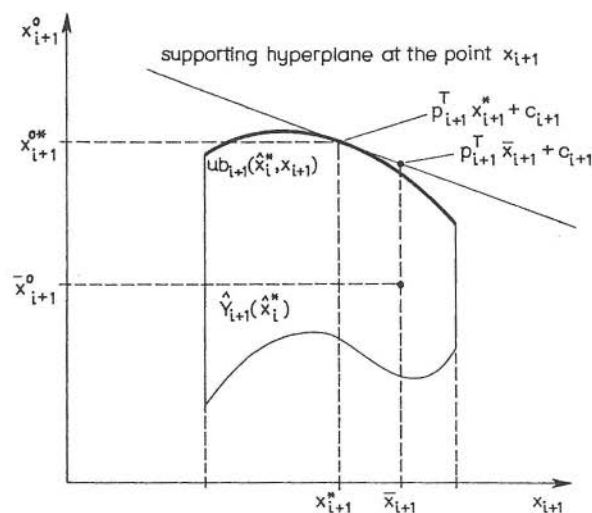


Fig. 3. Definition of the supporting hyperplane.

$$\bar{x}_{i+1}^0 \leq p_{i+1}^T \bar{x}_{i+1} + c_{i+1} \quad \text{for} \quad \hat{\bar{x}}_{i+1} = [\bar{x}_{i+1}^0, \bar{x}_{i+1}^T]^T \in \hat{Y}_{i+1}(\hat{x}_i^*), \quad (30b)$$

Moreover we have:

$$x_{i+1}^{0*} = x_i^{0*} + r_i(x_i^*, u_i^*), \quad (31a)$$

$$x_{i+1}^* = f_i(x_i^*, u_i^*), \quad (31b)$$

and for any $\bar{x}_{i+1} \in \hat{Y}_{i+1}(\hat{x}_i^*)$ there exists $\bar{u}_i \in U_i(x^*)$ such that:

$$\bar{x}_{i+1}^0 = x_i^{0*} + r_i(x_i^*, \bar{u}_i), \quad (32a)$$

$$\bar{x}_{i+1} = f_i(x_i^*, \bar{u}_i). \quad (32b)$$

Inserting (31) into (30a) and (32) to (30b) we get:

$$r_i(x_i^*, u_i^*) - p_{i+1}^T f_i(x_i^*, u_i^*) = c_{i+1} - x_i^{0*},$$

$$r_i(x_i^*, \bar{u}_i) - p_{i+1}^T f_i(x_i^*, \bar{u}_i) \leq c_{i+1} - x_i^{0*}.$$

Thus u_i^* maximizes the Hamiltonian (27) on $U_i(x_i^*)$.

Strict derivation of the maximum principle can be found e.g. in Canon at al. [6], Propoi [33, 34], or, under slightly different assumptions, in Vinter [45] in this issue.

A condition for existence of a supporting hyperplane — and therefore for validity of the maximum principle — was given by Halkin [17] who assumed that the sets $\hat{Y}_{i+1}(\hat{x}_i)$ were convex. This condition was shortly afterwards weakened by Holtzman [18] to the directional convexity of these

sets, i.e. to the concavity of the smaller upper boundary functions $ub_{i+1}(\hat{x}_i, \cdot)$. Because we have:

$$ub_{i+1}(\hat{x}_i^*, x_{i+1}^*) = UB_i(x_{i+1}^*),$$

$$ub_{i+1}(\hat{x}_i^*, x_{i+1}) \leq UB_i(x_{i+1}) \quad \text{for} \quad x_{i+1} \in Y_{i+1}(x_i^*),$$

then also convexity or directional convexity of the sets \hat{Y}_i can be used as a condition for validity of the maximum principle. The convexity of the set \hat{Y}_i was actually used by Propoi [32].

Thus besides the assumptions of differentiability of the problem functions, the existence of a finite optimal solution which satisfies the equation of motion, constraints, and a constraint qualification we need as sufficient conditions for validity of the maximum principle also concavity of the upper boundary functions. Conditions for that are given in Prop. A.1 in the Appendix. They are rather restrictive. At the same time it is not difficult to find real applications where the concavity does not hold, see e.g. [14, 27].

Nahorski et al. [27, 28] showed, for the case without constraints, that the directional concavity assumption can be substantially weakened in those problems where the greater upper boundary function is continuously differentiable. This idea will be now extended to the case with constraints in the form (24). It is not difficult to further extend it for the additional constraint of the form $u_i \in \Omega_i$, where Ω_i is a closed set.

When the smaller upper boundary function $ub_{i+1}(x_i^*, \cdot)$ is not concave we can use no supporting hyperplane. We often can, however, use a more complicated supporting surface whose equation is given by a nonlinear function:

$$x_{i+1}^0 = \pi_{i+1}(x_{i+1}) + c_{i+1}, \quad (33)$$

which satisfies:

$$x_{i+1}^{0*} = \pi_{i+1}(x_{i+1}^*) + c_{i+1}, \quad (34a)$$

$$x_{i+1}^0 \leq \pi_{i+1}(x_{i+1}) + c_{i+1} \quad \text{for} \quad \hat{x}_{i+1} \in \hat{Y}_{i+1}(\hat{x}_i^*), \quad (34b)$$

see also Fig. 4. If we now repeat the previous derivations we see that now u_i maximizes on $U_i(x_i^*)$ the following function:

$$NH_i(x_i^*, u_i, \pi_{i+1}) = r_i(x_i^*, u_i) - \pi_{i+1}[f_i(x_i^*, u_i)]. \quad (35)$$

We call $NH_i(x_i, u_i, \pi_{i+1})$ a nonlinear hamiltonian.

It is always possible to find a supporting surface π_{i+1} provided the smaller upper boundary $ub_{i+1}(x_i^*, \cdot)$ is finite on $Y_{i+1}(x_i^*)$. In the worst case it will be discontinuous. However, we restrict the class of supporting surfaces in order to derive the adjoint equation, i.e. the equation of the type (26b). Let us then suppose that besides the assumptions needed for the Karush-Kuhn-Tucker conditions to hold there exists a continuously

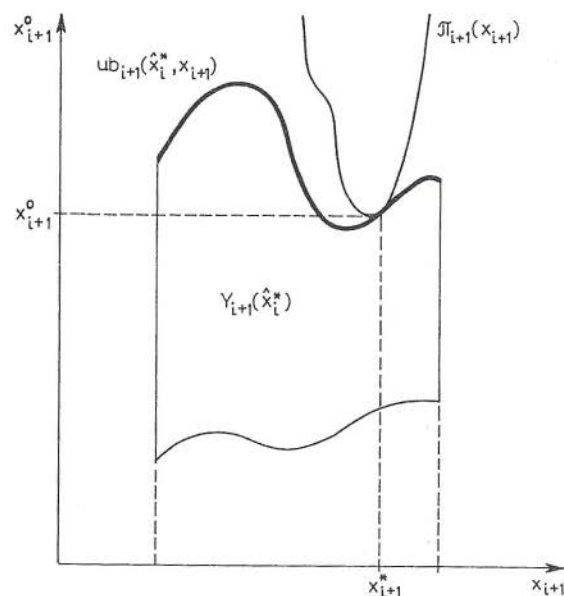


Fig. 4. Definition of the supporting surface.

differentiable supporting surface to the set $\hat{Y}_{i+1}(\hat{x}_i^*)$ at the point \hat{x}_{i+1}^* . When taken for all \hat{x}_{i+1} from the smaller upper boundary the latter assumption is weaker than that of the classical (linear hamiltonian) maximum principle directional concavity assumption. It is weaker than the assumption of the continuous differentiability of the function UB_{i+1} used by Nahorski et al. [27] in the unconstrained case, see Prop. A.9 in the Appendix for sufficient conditions. It is also weaker than Vinter's assumption [45] which is slightly stronger than the local directional concavity.

Referring to the Karush-Kuhn-Tucker conditions it can be shown that this supporting surface can be chosen to satisfy:

$$\nabla_x \pi_{i+1}(x_{i+1}^*) = p_{i+1}^T. \quad (36)$$

The intuitive idea behind it is that either UB_i is concave at x_{i+1}^* , at least locally, or it is differentiable. It is evident that π_{i+1} can be chosen to satisfy the above equality in the former case. In the latter p_{i+1}^T is the derivative of UB_i at x_{i+1}^* and thus the equality is also true.

Now, under (36) we have:

$$\nabla_x NH_i(x_i^*, u_i^*, \pi_{i+1}) = \nabla_x H_i(x_i^*, u_i^*, p_{i+1}). \quad (37)$$

Inserting $\nabla_x NH_i$ for $\nabla_x H_i$ and $\nabla_x \pi_i(x_i^*)$ for p_i^T in (26b) we get the generalized adjoint equation:

$$-\nabla_x \pi_i(x_i^*) = \nabla_x \text{NH}_i(x_i^*, u_i^*, \pi_{i+1}) - \mu_i^T \nabla_x h_i(x_i^*, u_i^*) - \lambda_i^T \nabla_x g_i(x_i^*, u_i^*) \quad (38a)$$

$$\lambda_i^T g_i(x_i^*, u_i^*) = 0, \quad (38b)$$

$$\lambda_i \geq 0, \quad (38c)$$

for $i = 0, 1, \dots, N-1$, with $\pi_0(x_0) \equiv 0$, and:

$$\nabla_x \pi_N(x_N^*) = \mu_N^T \nabla_x h_N(x_N^*) + \lambda_N \nabla_x g_N(x_N^*), \quad (38d)$$

$$\lambda_N^T g_N(x_N^*) = 0, \quad (38e)$$

$$\lambda_N \geq 0. \quad (38f)$$

The above conditions together with the nonlinear hamiltonian maximization:

$$\sup_{u_i \in U_i(x_i^*)} \text{NH}_i(x_i^*, u_i, \pi_{i+1}), \quad (38g)$$

form the constrained generalized maximum principle. Needless to say that when $\pi_{i+1}(x_{i+1}) = p_{i+1}^T x_{i+1}$, i.e. π_{i+1} is linear, then the generalized maximum principle reduces to the classical one.

Quite recently the problem (1) with the nondifferentiable problem functions has been studied. This approach uses the Clarke generalized gradient and based on it the Clarke multiplier rule [7, 8, 9]. They are basically defined for Lipschitz continuous functions. In our case it may be necessary to use this approach even when all the problem functions are differentiable but a Lipschitz continuous supporting surface has to be taken to make sure that the hamiltonian is maximized.

This direction of research was probably started by Doležal [11, 12] who derived the stationary conditions analogous to (26). Studniarski [43] and Pytlak & Malinowski [35] considered some special cases. Vinter [45] in this issue gives the maximum principle analogous to the classical (linear hamiltonian) one.

Taking the same assumptions as previously, except that the differentiability of the problem functions is now weakened to only Lipschitz continuity, it follows from the Clarke multiplier rule that there exist multipliers μ_i , λ_i such that the following conditions hold:

$$0 \in \partial_x [\text{NH}_i(x_i^*, u_i^*, \pi_{i+1}) + \pi_i(x_i^*) - \mu_i^T h_i(x_i^*, u_i^*) - \lambda_i^T g_i(x_i^*, u_i^*)], \quad (39a)$$

$$\lambda_i^T g_i(x_i^*, u_i^*) = 0, \quad (39b)$$

$$\lambda_i \geq 0 \quad (39c)$$

Above, ∂_x denotes the Clarke generalized gradient with respect to x which for a Lipschitz continuous function $t(x, u)$ can be defined as the convex hull, denoted by co , of the limits of gradients taken in those points where they exist:

$$\partial_x t(x, u) = \text{co} \lim_{i \rightarrow \infty} \{\nabla_x t(x_i, u) : x_i \rightarrow x\}. \quad (40)$$

If π_i , $i = 0, 1, \dots, N$ is continuously differentiable (or only strictly differentiable) which is possible even when the problem functions are only Lipschitz continuous, then (39b) can be given in the form:

$$-\nabla_x \pi_i(x_i^*) \in \partial_x [NH_i(x_i^*, u_i^*, \pi_{i+1}) - \mu_i^T h_i(x_i^*, u_i^*) - \lambda_i^T g_i(x_i^*, u_i^*)], \quad (39b')$$

while for π_i , π_{i+1} only Lipschitz continuous but the problem functions differentiable:

$$\mu_i^T h_i(x_i^*, u_i^*) + \lambda_i^T g_i(x_i^*, u_i^*) \in \partial_x [NH_i(x_i^*, u_i^*, \pi_{i+1}) + \pi_i(x_i^*)]. \quad (39b'')$$

The conditions (39) together with the condition for maximization of the nonlinear hamiltonian (35) form the nonsmooth constrained generalized maximum principle.

4. The upper boundary and dynamic programming

Dynamic programming is a method of directly working with the (reverse) greater upper boundaries called there the (optimal) value or return functions. It uses the recursive dependence:

$$RUB_i(x_i) = \sup \{r_i(x_i, u_i) + RUB_{i+1}[f_i(x_i, u_i)]\}, \quad (41a)$$

over all u_i satisfying:

$$u_i \in U_i(x_i), \quad (41b)$$

$$f_i(x_i, u_i) \in X_{i+1}, \quad (41c)$$

for $x_i \in X_i$, $i = N-1, N-2, \dots, 0$, with:

$$RUB_N(x_N) = 0. \quad (41d)$$

Let us transform the equation (41a) to the form:

$$\sup \{r_i(x_i, u_i) - RUB_i(x_i) + RUB_{i+1}[f_i(x_i, u_i)]\} = 0. \quad (42)$$

We see from (22) and (23) that $-RUB_i$ satisfies the formulae (34) and therefore $-RUB_i$ is a supporting surface to the set \hat{Y}_{i+1} and thus also to $\hat{Y}_{i+1}(\hat{x}_i^*)$ at the point $\hat{x}_{i+1}^* = [x_i^* + r_i(x_i^*, u_i^*), f_i^T(x_i^*, u_i^*)]^T$. Therefore (42) subject to (41b-c) can be compared to the maximization of the hamiltonian (38g). The difference in the set of constraints which is smaller in (41) does not influence the optimal solution because it obviously is situated in the region given by (41b-c) for $x_i = x_i^*$.

The above observation can help in finding the optimal solution. Some possibilities of doing this have been shown by Vidal [44].

Unfortunately, application of dynamic programming is rather limited due to the so called "curse of dimensionality". This was the reason for its further development which would make it possible to solve larger problems. It was done for sufficiently smooth functions.

The idea is to approximate locally the reverse greater upper boundary function by a simpler one like linear or quadratic. It can be traced back to the book by Bellman & Dreyfus [2]. Its computational techniques were mainly founded by Jacobson & Mayne [20] where the name "differential dynamic programming" was coined. The numerical algorithms were also discussed by McReynolds [26] and Dyer & McReynolds [13] who used quadratic approximations. This idea was further developed by Ohno [29] and Yakowitz [46, 47], see also a survey paper by Yakowitz & Rutherford [48]. Twice or thrice differentiable greater upper boundary functions were required. Conditions for higher order differentiability of the upper boundary functions are given in Prop. A.11 in the Appendix.

We sketch this approach for a quadratic approximation. Let (\bar{x}_i, \bar{u}_i) , $i = 0, 1, \dots, N-1$, \bar{x}_N be any admissible solution to the problem (1). Let us assume that the functions are thrice differentiable around the point (\bar{x}_i, \bar{u}_i) . Then the function $r_i(x_i, u_i) + \text{RUB}_{i+1}[f_i(x_i, u_i)]$, see (41), can be expanded in the second order Taylor series around (\bar{x}_i, \bar{u}_i) . Let us denote this second order approximation by $Q_i(x_i, u_i)$. It can be used to find an approximate optimal dependence $u_i^*(x_i)$. This is typically done by solving the (stagewise) Karush-Kuhn-Tucker conditions [29, 46]. For that $f_i(x_i, u_i)$ can be linearly approximated to simplify the task. Inserting $u_i^*(x_i)$ for u_i in $Q_i(x_i, u_i)$ an approximation to RUB_i around \bar{x}_i is obtained. It allows to compute the approximation $Q_{i-1}(x_{i-1}, u_{i-1})$. Repeating the calculations we can approximate locally all the reverse upper boundary functions RUB_i and find $u_i^*(x_i)$, $i = N-1, N-2, \dots, 0$. Now, using (21) and then recursively, $i = 0, \dots, N-1$, (1b) and $u_i^*(x_i)$ a better approximation to an optimal solution can be found.

We see that an approximation to the reverse upper boundary functions is instrumental in the method.

An important step in the derivation is the use of the stagewise Karush-Kuhn-Tucker conditions, connected with the problem (41a-b). Yakowitz [46] shows that if the stagewise Karush-Kuhn-Tucker conditions are satisfied then the Karush-Kuhn-Tucker conditions for the entire problem are also satisfied. Here we give a different derivation of this result.

Let us consider a stagewise (for a fixed i) problem (41a-b). Assume that the functions $r_i, f_i, g_i, h_i, \text{RUB}_i, \text{RUB}_{i+1}$ are continuously differentiable near a point (\bar{x}_i, \bar{u}_i) at which there exist $p_{i+1}, \lambda_i, \mu_i$ such that the following Karush-Kuhn-Tucker conditions hold:

$$p_N^T - \mu_N^T \nabla_x h_N - \lambda_N^T \nabla_x g_N = 0, \quad (43a)$$

if $i = N$;

$$\nabla_u r_{N-1} - \mu_{N-1}^T \nabla_u h_{N-1} - \lambda_{N-1}^T \nabla_u g_{N-1} - p_N^T \nabla_u f_{N-1} = 0, \quad (43b)$$

if $i = N-1$;

$$\nabla_u r_i + \nabla_x \text{RUB}_{i+1} \nabla_u f_i - \lambda_i^T \nabla_u h_i - \lambda_i^T \nabla_u g_i = 0, \quad (43c)$$

if $i \leq N-2$;

$$\lambda_i \geq 0, \quad (43d)$$

$$\lambda_i^T g_i = 0, \quad (43e)$$

for any i . For simplicity the arguments (\bar{x}_i, \bar{u}_i) above were dropped.

Let us additionally assume that the point-to-set map $U_i(x_i) \neq \emptyset$ is uniformly compact near \bar{x}_i , see Def. A.3 in the Appendix. Dropping again the arguments we have, see Prop. A.9 in the Appendix:

$$\nabla_x \text{RUB}_{N-1} = \nabla_x r_{N-1} - \mu_{N-1}^T \nabla_x h_{N-1} - \lambda_{N-1}^T \nabla_x g_{N-1} - p_N^T \nabla_x f_{N-1} \quad (44a)$$

if $i = N-1$, and

$$\nabla_x \text{RUB}_i = \nabla_x r_i + \nabla_x \text{RUB}_{i+1} \nabla_x f_i - \mu_i^T \nabla_x h_i - \lambda_i^T \nabla_x g_i \quad (44b)$$

if $i \leq N-2$.

Defining now $\nabla_x \text{RUB}_i(\bar{x}_i) = -p_i^T$ and taking (43)–(44) for $i = 0, 1, \dots, N-1$ we see that they form the Karush-Kuhn-Tucker conditions (26) for the entire problem.

It is evident that among the points (\bar{x}_i, \bar{u}_i) satisfying (43) there exists a sequence (\bar{x}_i, \bar{u}_i) , $i = 0, 1, \dots, N-1$, \bar{x}_N which satisfies the equation of motion (1b). This is, of course, any optimal sequence which satisfies (25) and thus (26). However, we additionally assumed in (44) that the point-to-set map $U_i(x_i) \neq \emptyset$ be uniformly compact near \bar{x}_i . This assumption which excludes some pathological cases will not be further discussed here. Note, however, that Yakowitz [46] uses instead an assumption of continuous differentiability of the function $u_i^*(x_i)$.

5. The upper boundary and the Krotov conditions

Let us reformulate the problem (1) as follows:

$$\begin{aligned} \text{maximize } J = r_0(x_0, u_0) - \pi_1(x_1) + \sum_{i=1}^{N-1} [r_i(x_i, u_i) + \\ + \pi_i(x_i) - \pi_{i+1}(x_{i+1})] + \pi_N(x_N), \end{aligned} \quad (45)$$

subject to (1b–d). Inserting (1b) for x_{i+1} in $\pi_{i+1}(x_{i+1})$ and dropping the restriction (1b) we get the relaxed problem:

$$\begin{aligned} \text{maximize } I = r_0(x_0, u_0) - \pi_1[f_0(x_0, u_0)] + \sum_{i=1}^{N-1} \{r_i(x_i, u_i) + \\ + \pi_i(x_i) - \pi_{i+1}[f_i(x_i, u_i)]\} + \pi_N[f_{N-1}(x_{N-1}, u_{N-1})], \end{aligned} \quad (46a)$$

subject to:

$$(x_i, u_i) \in V_i \quad \text{for } i = 0, 1, \dots, N-1, \quad (46b)$$

$$x_N \in V_N. \quad (46c)$$

Contrary to the problem (1) the problem (46) is decomposable stagewise. Any optimal solution (x_i^*, u_i^*) , $i = 0, 1, \dots, N-1$, x_N^* to the problem (46) is the same as to the problem (1) provided $f_i(x_i^*, u_i^*) = x_{i+1}^*$. This results from the Krotov principle of extensions [23, 24, 25].

Krotov [23] proved that if (x_i, u_i) , $i = 0, 1, \dots, N-1$, x_N satisfy (1b), then a sufficient, and in the case when $r_i(x_i, u_i)$, $i = 0, 1, \dots, N-1$ are bounded on V_i also a necessary condition for (1) to hold is that there exist π_i , $i = 1, 2, \dots, N$ such that the following exist and are finite:

$$\sup_{(x_i, u_i) \in V_i} C_i[x_i, f_i(x_i, u_i), u_i, \pi_i, \pi_{i+1}] < \infty, \quad (47a)$$

$$\begin{aligned} i &= 0, 1, \dots, N-1 \\ \sup_{x_N \in V_N} C_N(x_N) &< \infty, \end{aligned} \quad (47b)$$

where:

$$C_i(x_i, x_{i+1}, u_i, \pi_i, \pi_{i+1}) = r_i(x_i, u_i) + \pi_i(x_i) - \pi_{i+1}(x_{i+1}), \quad (48a)$$

$$\begin{aligned} i &= 0, 1, \dots, N-1 \\ \pi_0(x_0) &= 0, \end{aligned} \quad (48b)$$

$$C_N(x_N) = \pi_N(x_N). \quad (48c)$$

Comparing (48a) with (42) we see that if we let $\pi_i(x_i) = -\text{RUB}_i(x_i)$ we get almost identical expressions. The only difference is that π_i need to be defined on a larger set than RUB_i . Thus, letting $Z_i = \{x_i | \exists (x_{i-1}, u_{i-1}) \in V_{i-1} : x_i = f_{i-1}(x_{i-1}, u_{i-1})\}$ we see that π_i need be defined on the set $X_i \cup Z_i$, while RUB_i is defined on the set RY_i . Since in general $\text{RY}_i \subseteq X_i$ we see that $\text{RY}_i \subseteq X_i \cup Z_i$. If r_i is bounded on V_i then we can define $\pi_i(x_i) = -\text{RUB}_i(x_i)$ for $x_i \in \text{RY}_i$ and $\pi_i(x_i)$ sufficiently small outside RY_i . Similarly we could have chosen $\pi_i(x_i)$ identical to $\text{UB}_i(x_i)$ on Y_i and sufficiently small otherwise. In both cases we would then find that with this choice of π the optimal (x_i^*, u_i^*) in problem (1) will be maximizing in (47)–(48).

It is not difficult to see that when (1b) is satisfied, then the Krotov functions π_i are actually supporting surfaces to the sets \hat{Y}_i . Indeed, let (47) hold. Because the sequence (x_j, u_j) , $j = 0, 1, \dots, i$ satisfies (1b) we have:

$$\begin{aligned} \sum_{j=0}^i \sup_{(x_j, u_j) \in V_j} C_j[x_j, f_j(x_j, u_j), u_j, \pi_j, \pi_{j+1}] &= \\ &= \sup_{\substack{(x_j, u_j) \in V_j \\ f_j(x_j, u_j) = x_{j+1}}} \sum_{j=0}^i C_j[x_j, f_j(x_j, u_j), u_j, \pi_j, \pi_{j+1}] = \\ &= \sup_{\substack{(x_j, u_j) \in V_j \\ f_j(x_j, u_j) = x_{j+1}}} \sum_{j=0}^i r_j(x_j, u_j) - \pi_{i+1}(x_{i+1}) = c_{i+1} < \infty. \end{aligned} \quad (49)$$

Let us assume that \hat{Y}_{i+1} is closed (if not — we can consider the closure). Then from (49) there exist a \hat{x}_{i+1}^A such that:

$$x_{i+1}^{0A} = \pi_{i+1}(x_{i+1}^A) + c_{i+1}. \quad (50a)$$

We also have:

$$x_{i+1}^0 \leq \pi(x_{i+1}) + c_{i+1} \quad \text{for all } \hat{x}_{i+1} \in \hat{Y}_{i+1}, \quad (50b)$$

and therefore π_{i+1} is a supporting surface to \hat{Y}_{i+1} at \hat{x}_{i+1}^A . Vice versa, let π_{i+1} satisfy (50) and (x_j, u_j) , $j = 0, 1, \dots, i$ satisfy (1b). Then arguing backwards we can show that (47) holds.

Similarly the functions $-\pi_i$ can be shown to be supporting surfaces to the sets $R\hat{Y}_i$.

These conditions of support impose restrictions on the choice of π , in order that the solution to (1) shall also solve (47)–(48). We can give sufficient conditions for π in the following way, cf. Ravn [37]. Define on the set X_{i+1} :

$$F_{i+1}(x_{i+1}) = \sup [\pi_i(x_i) + r_i(x_i, u_i)], \quad (51)$$

subject to (1b)–(1c). Similarly define on the set X_i :

$$RF_i(x_i) = \sup [\pi_{i+1}(f_i(x_i, u_i)) + r_i(x_i, u_i)], \quad (52)$$

subject to (1c). Then the fulfillment of the following conditions is necessary and sufficient for the solution of (1) to be also a solution to (47)–(48):

$$UB_i(x_i) - UB_i(x_i^*) \leq F_i(x_i) - F_i(x_i^*), \quad \forall x_i \in X_i \cap Y_i, \quad (53)$$

$$F_i(x_i) - F_i(x_i^*) \leq \pi_i(x_i) - \pi_i(x_i^*), \quad \forall x_i \in X_i, \quad (54)$$

$$\pi_i(x_i) - \pi_i(x_i^*) \leq -RF_i(x_i) + RF_i(x_i^*), \quad \forall x_i \in X_i, \quad (55)$$

$$-RF_i(x_i) + RF_i(x_i^*) \leq -RUB_i(x_i) + RUB_i(x_i^*), \quad \forall x_i \in X_i \cap Y_i. \quad (56)$$

We see that these conditions link π_i to UB_i and RUB_i , and also, via (51)–(52) link π_i with π_{i-1} and π_{i+1} .

An important assumption in the Krotov necessary and sufficient conditions is that (1b) is satisfied. This can be abandoned considering instead the problem:

$$\sup_G \inf_{\Pi} \left\{ \sum_{i=0}^{N-1} C_i[x_i, f_i(x_i, u_i), u_i, \pi_i, \pi_{i+1}] + C_N x_N \right\} \quad (57)$$

where G is the set of (x_i, u_i) , $i = 0, 1, \dots, N-1$, x_N satisfying (1c–d) and Π is a subset of the set of all π_i , $i = 1, 2, \dots, N$ such that:

$$\inf [\pi_i(a) - \pi_i(b)] = -\infty \quad \text{if } a \neq b. \quad (58)$$

The above condition forces in (57) satisfaction of (1b). The problem (57) together with the dual problem:

$$\inf_{\Pi} \sup_G \left\{ \sum_{i=0}^{N-1} C_i [x_i, f_i(x_i, u_i), u_i, \pi_i, \pi_{i+1}] + C_N(x_N) \right\}, \quad (59)$$

was discussed by Ravn [36]. This pair of dual problems is unfortunately not decomposable stagewise.

In general the weak duality relationship holds:

$$\begin{aligned} \sup_G \inf_{\Pi} \left[\sum_{i=0}^{N-1} C_i(x_i, f_i(x_i, u_i), u_i, \pi_i, \pi_{i+1}) + C_N(x_N) \right] &\leq \\ &\leq \inf_{\Pi} \sup_G \left[\sum_{i=0}^{N-1} C_i(x_i, f_i(x_i, u_i), u_i, \pi_i, \pi_{i+1}) + C_N(x_N) \right], \end{aligned} \quad (60)$$

and the solution of (60) therefore provides an upper bound on the optimal criterion value in (1). If there is no restriction on the choice of π , other than (58), and if the problem (1) has a solution, we can get equality in (60), i.e. the dual gap vanishes. Under appropriate choice of Π algorithms can be specified, which are based on the solution of the dual problem (59).

In Rockafellar [42] in this issue the case of a convex problem (1) and linear π is discussed.

6. Conclusions

The goal of this paper is twofold. Firstly, we generalized to the state constrained case the geometrical concepts of [27] which form the basis of the upper boundary approach. Secondly, the connections of this approach to other main methods of solving the multistage optimization problems were discussed. We showed that either explicitly or implicitly all these methods work with the upper boundary functions. We also formulated and motivated the generalized maximum principle for the mixed state and control constrained case which was earlier proved for the unconstrained case in [27].

We think that due to the easy geometrical interpretation the upper boundary approach is suitable for lecturing purposes. We hope also that it can contribute to creating new or improving existing algorithms for solving multistage optimization problems.

7. Appendix. Properties of the upper boundary functions

Let X be a subset of an Euclidean space.

DEFINITION A.1.

A function $f: X \rightarrow R$ is called upper semicontinuous at a point x_0 if, for all $\varepsilon > 0$, there exists a neighbourhood N_{x_0} of x_0 such that:

$$x \in N_{x_0} \Rightarrow f(x) < f(x_0) + \varepsilon. \quad (\text{A.1})$$

DEFINITION A.2. A point-to-set map $G: X \rightarrow 2^{R^m}$ is said to be upper semi-continuous (u.s.c.) at $x_0 \in X$ if for all open sets $S \subseteq R^m$ containing $G(x_0)$ there exists a neighbourhood N_{x_0} such that:

$$x \in N_{x_0} \Rightarrow G(x) \subseteq S. \quad (\text{A.2})$$

DEFINITION A.3. A point-to-set map $G: X \rightarrow 2^{R^m}$ is said to be uniformly compact near x_0 if there exists a neighbourhood N_{x_0} of x_0 such that the closure of the set $B = \bigcup_{x \in N_{x_0}} G(x)$ is compact.

In Euclidean spaces (which are actually considered in this paper) instead of the above condition of uniform compactness a condition of uniform boundedness can be equivalently used. Then the set B is bounded.

Still other conditions for some kind of stability of G have been introduced. E.g. Rockafellar's [40] tameness is a condition which is weaker than uniform compactness.

A sequence of controls $\mathcal{U}_i = \{u_0, u_1, \dots, u_i\}$, $0 \leq i \leq N-1$, will be called a *strategy (to stage i)*. A resulting sequence of states (depending on x_0) $\chi_i(x_0) = \{x_0, x_1, \dots, x_i\}$ where $x_{j+1} = f_j(x_j, u_j)$, $u_j \in \mathcal{U}_i$, $j = 0, 1, \dots, i-1$, will be called a *trajectory (to stage i)*. A strategy \mathcal{U}_i such that all its elements together with the states at the same stages satisfy the constraints, i.e. $(x_j, u_j) \in V_j$, $u_j \in \mathcal{U}_i$, $x_j \in \chi_i(x_0)$, $x_0 \in X_0$, $j = 0, 1, \dots, i$, will be called an *admissible strategy (to stage i)* and the resulting trajectory an *admissible trajectory (to stage i)*. All such pairs form the set of all admissible strategies (to stage i) called \mathcal{S}_i and all admissible trajectories (to stage i) called \mathcal{T}_i .

A sequence of controls $\mathcal{H}_i = \{u_i, u_{i+1}, \dots, u_N\}$, $0 \leq i \leq N-1$, will be called a *terminating strategy (from stage i)*. A resulting sequence of states, which depends on x_i , $\mathcal{H}\chi_{i+1}(x_i) = \{x_i, x_{i+1}, \dots, x_N\}$, where $x_{j+1} = f_j(x_j, u_j)$, $u_j \in \mathcal{H}_i$, $j = i, i+1, \dots, N-1$, will be called a *terminating strategy (from stage i)*. A terminating strategy such that each of its elements together with the state at the same stage satisfy the constraint (x_j, u_j) , $j = i, i+1, \dots, N-1$, $x_N \in V_N$ will be called an *admissible terminating strategy (from stage i)* and an associated trajectory an *admissible terminating trajectory (from stage i)*. All such pairs form the sets of admissible terminating strategies (from stage i) called $\mathcal{H}\mathcal{S}_i$ and admissible terminating trajectories (from stage i) called $\mathcal{H}\mathcal{T}_i$.

We define the following point-to-set maps. The point-to-set map of admissible controls which relate the states $x_i \in X_i$ and $x_{i+1} \in X_{i+1}$:

$$G_i^s: X_i \times X_{i+1} \rightarrow 2^{R^m}, G_i^s(x_i, x_{i+1}) = S_{x_i}[f_i^{-1}(x_{i+1})] \cap U_i(x_i), \quad (\text{A.3})$$

the point-to-set map of optimal controls which move the state from $x_i \in Y_i$ to $x_{i+1} \in Y_{i+1}(x_i)$:

$$\begin{aligned} O_i^s: Y_i \times Y_{i+1}(x_i) &\rightarrow 2^{R^m}, O_i^s(x_i, x_{i+1}) = \\ &= \{u_i \in G_i^s(x_i, x_{i+1}): x_i^0 + r_i(x_i, u_i) = \text{ub}_{i+1}(\hat{x}_i, x_{i+1})\}, \end{aligned} \quad (\text{A.4})$$

the point-to-set map of optimal controls which allow to reach $x_{i+1} \in \text{RY}_{i+1}$ from $x_i \in \text{RY}_i(x_{i+1})$:

$$\begin{aligned} \text{RO}_i^s: \text{RY}_{i+1} \times \text{RY}_i(x_{i+1}) &\rightarrow 2^{R^m}, \text{RP}_i^s(x_{i+1}, x_i) = \\ &= \{u_i \in G_i^s(x_i, x_{i+1}): x_{i+1}^\square + r_i(x_i, u_i) = \text{rub}_i(x_{i+1}^\square, x_i)\}, \end{aligned} \quad (\text{A.5})$$

the point-to-set map of admissible strategies (to stage i) moving an initial state $x_0 \in X_0$ to $x_{i+1} \in Y_{i+1}$:

$$\begin{aligned} \mathcal{G}_i^g: Y_{i+1} &\rightarrow \mathcal{S}_i, \mathcal{G}_i^g(x_{i+1}) = \\ &= \{\mathcal{U}_i \in \mathcal{S}_i: \exists x_0 \in X_0, \chi(x_0, \mathcal{U}_i) \in \mathcal{T}_i, f_i(x_i, u_i) = x_{i+1}\} \end{aligned} \quad (\text{A.6})$$

the point-to-set map of optimal strategies (to stage i) moving an initial state $x_0 = X_0$ to $x_{i+1} \in Y_{i+1}$:

$$\begin{aligned} \mathcal{C}_i^g: Y_{i+1} &\rightarrow \mathcal{S}_i, \mathcal{C}_i^g(x_{i+1}) = \{\mathcal{U}_i \in \mathcal{G}_i^g(x_{i+1}): \sum_{j=0}^i r_j(x_j, u_j) = \\ &= \text{UB}_{i+1}(x_{i+1})\} \end{aligned} \quad (\text{A.7})$$

the point-to-set map of terminating strategies (from stage i) moving a $x_i \in \text{RY}_i$ to an end state $x_N \in X_N$:

$$\mathcal{R}\mathcal{G}_i^g: \text{RY}_i \rightarrow \mathcal{R}\mathcal{S}_i, \mathcal{R}\mathcal{G}_i^g(x_i) = \{\mathcal{R}\mathcal{U}_i \in \mathcal{R}\mathcal{S}_i: \mathcal{R}\chi_i(x_i, \mathcal{R}\mathcal{U}_i) \in \mathcal{R}\mathcal{T}_i\} \quad (\text{A.8})$$

the point-to-set map of optimal terminating strategies (from stage i):

$$\begin{aligned} \mathcal{R}\mathcal{C}_i^g: \text{RY}_i &\rightarrow \mathcal{R}\mathcal{S}_i, \mathcal{R}\mathcal{C}_i^g(x_i) = \{\mathcal{R}\mathcal{U}_i \in \mathcal{R}\mathcal{G}_i^g(x_i): \sum_{j=i}^{N-1} r_j(x_j, u_j) = \\ &= \text{RUB}_i(x_i)\} \end{aligned} \quad (\text{A.9})$$

For the case with V_i specified as in (24) we introduce the stagewise Lagrangean:

$$\begin{aligned} L_i &= \lambda_i^0 r_i(x_i, u_i) - p_{i+1}^T [f_i(x_i, u_i) - x_{i+1}] - \\ &\quad - \mu_i^T h_i(x_i, u_i) - \lambda_i^T g_i(x_i, u_i), \end{aligned} \quad (\text{A.10a})$$

$$L_N = -\mu_N^T h_N(x_N) - \lambda_N^T g_N(x_N) \quad (\text{A.10b})$$

and the partial Lagrangean:

$$\begin{aligned} L_i^p &= \sum_{j=0}^i \lambda_i^0 r_j(x_j, u_j) - p_{i,j+1}^T [f_j(x_j, u_j) - x_{j+1}] - \mu_{ij}^T h_j(x_j, u_j) - \\ &\quad - \lambda_{ij}^T g_j(x_j, u_j) = \sum_{j=0}^i L_{ij} \quad \text{for } i = 0, 1, \dots, N-1 \end{aligned} \quad (\text{A.11a})$$

$$L_N^p = \sum_{j=0}^{N-1} L_{Nj} - \mu_{NN}^T h_N(x_N) - \lambda_{NN}^T g_N(x_N), \quad (\text{A.11b})$$

where:

$$L_{ij} = \lambda_i^0 r_j(x_j, u_j) - p_{i,j+1}^T [f_j(x_j, u_j) - x_{j+1}] - \mu_{ij}^T h_j(x_j, u_j) - \lambda_{ij}^T g_j(x_j, u_j). \quad (\text{A.12})$$

Under the continuous differentiability assumption if, for a given x_i, u_i maximizes on $U_i(x_i)$ the function $r_i(x_i, u_i)$ subject to $f_j(x_j, u_j) = x_{j+1}$ for a given x_{i+1} (i.e. on a set defined by (A.3)), then the following Fritz John conditions hold:

$$\lambda_i^0 \nabla_u r_i(x_i, u_i) - p_{i,i+1}^T \nabla_u f_i(x_i, u_i) - \mu_i^T \nabla_u h_i(x_i, u_i) - \lambda_i^T \nabla_u g_i(x_i, u_i) = 0 \quad (\text{A.13a})$$

$$\lambda_i^T g_i(x_i, u_i) = 0, \quad (\text{A.13b})$$

$$\lambda_i \geq 0, \quad (\lambda_i^0, p_{i,i+1}^T, \mu_i^T, \lambda_i^T) \neq 0. \quad (\text{A.13c})$$

If u_i, x_i belong to the optimal strategy and resulting trajectory (both to stage i), then there hold:

$$\lambda_i^0 \nabla_u r_i(x_i, u_i) - p_{i,i+1}^T \nabla_u f_i(x_i, u_i) - \mu_{ii}^T \nabla_u h_i(x_i, u_i) - \lambda_{ii}^T \nabla_u g_i(x_i, u_i) = 0 \quad (\text{A.14a})$$

$$\lambda_{ii}^T g_i(x_i, u_i) = 0, \quad (\text{A.14b})$$

$$\lambda_{ii} \geq 0, \quad (\lambda_i^0, p_{i,i+1}^T, \mu_{ii}^T, \lambda_{ii}^T) \neq 0. \quad (\text{A.14c})$$

If the problem functions are not continuously differentiable, then Fritz John conditions can be formulated using the Clarke's generalized gradient, compare (40) and (39). This will permit the same conclusions with respect to Lagrange multipliers, defined above.

Let us introduce the following subsets of Lagrange multipliers $\Lambda_i = (\lambda_i^0, \mu_i^T, \lambda_i^T, p_{i,i+1}^T)$ associated with the Lagrangean (A.10):

$$K_{ui}^0 = \{\Lambda_i : (0, \mu_i^T, \lambda_i^T, p_{i,i+1}^T) \text{ satisfy the Fritz John necessary conditions (A.13) for a given } u_i\}$$

$$K_{ui}^2 = \{\Lambda_i : (1, \mu_i^T, \lambda_i^T, p_{i,i+1}^T) \text{ satisfy the Fritz John necessary conditions (A.13) for a given } u_i, \text{ i.e. they satisfy the Karush-Kuhn-Tucker necessary conditions}\}$$

and the subsets of Lagrange multipliers $\Lambda_{ii}^T = (\lambda_i^0, \mu_{ii}^T, \lambda_{ii}^T, p_{i,i+1}^T)$ associated with the Lagrangean (A.11):

$$K_{uii}^0 = \{\Lambda_{ii} : (0, \mu_{ii}^T, \lambda_{ii}^T, p_{i,i+1}^T) \text{ satisfy the Fritz John necessary conditions (A.14) for a given } u_i\}$$

$$K_{uii}^1 = \{\Lambda_{ii} : (1, \mu_{ii}^T, \lambda_{ii}^T, p_{i,i+1}^T) \text{ satisfy the Fritz John necessary conditions (A.14) for a given } u_i, \text{ i.e. they satisfy the Karush-Kuhn-Tucker necessary conditions}\}.$$

DEFINITION A.4. Let us suppose that the functions g_i , h_i , and f_i are continuously differentiable on W_i . Then they satisfy the Mangasarian-Fromowitz constraint qualification at a feasible point $u_i \in G_i^s(x_i, x_{i+1})$ if:

(i) there exists $y_i \in R^m$ such that:

$$\begin{aligned} \nabla_u h_i(x_i, u_i) y_i &= 0 \\ \nabla_u f_i(x_i, u_i) y_i &= 0 \\ \nabla_u g_i(x_i, u_i) y_i &< 0 \quad \text{for } j \in I_i \\ I_i &= \{j: g_i^j(x_i, u_i) = 0\} \end{aligned} \quad (\text{A.15a})$$

(ii) the rows of the Jacobian:

$$\begin{bmatrix} \nabla_u h_i(x_i, u_i) \\ \nabla_u f_i(x_i, u_i) \end{bmatrix} \quad (\text{A.15b})$$

are linearly independent.

Satisfaction of the Mangasarian-Fromowitz constraint qualification is equivalent to each of the following conditions, see [41]:

- (i) $K_{ui}^0 = \emptyset$ ($K_{uii}^0 = \emptyset$)
- (ii) K_{ui}^1 (K_{uii}^1) is nonempty and bounded.

DEFINITION A.5. Under the assumptions of Def. A.4 the functions satisfy the strong constraint qualification at a feasible point $u_i \in G_i^s(x_i, x_{i+1})$ if the rows of the following matrix:

$$\begin{bmatrix} \nabla_u h_i(x_i, u_i) \\ \nabla_u f_i(x_i, u_i) \\ \nabla_u g_i^j(x_i, u_i) \end{bmatrix}_{j \in I_i} \quad (\text{A.16})$$

are linearly independent.

Satisfaction of the strong constraint qualification is equivalent to the following condition:

- (i) K_{ui}^1 (K_{uii}^1) consists of only one point (i.e. the Lagrange multipliers are unique)

Another constraint qualification which is not used in this paper is the calmness introduced by Clarke [8]. Its effect on the sets K_{ui}^0 (K_{uii}^0) and K_{ui}^1 (K_{uii}^1) is close to the effect of the Mangasarian-Fromowitz qualification.

PROPOSITION A.1. If r_i is concave on the set W_i , g_i^j , $j = 1, \dots, k_i$, are quasiconvex on W_i , h_i^j , $j = 1, \dots, l_i$, and f_i^j , $j = 1, \dots, n$, are quasimonotonic on W_i , and $Y_{i+1}(x_i)$ [$RY_i(x_{i+1})$] is a convex set, then $\text{ub}_{i+1}(\hat{x}_i, \cdot)$ [$\text{rub}_i(\tilde{x}_{i+1}, \cdot)$] is concave on $Y_{i+1}(x_i)$ [$RY_i(x_{i+1})$].

If the above assumptions on the functions r_i , g_i , h_i , and f_i are satisfied for all $i = 0, 1, \dots, N-1$ and N when appropriate, and Y_{i+1} [RY_i] are convex sets, then UB_{i+1} [RUB_i] are concave on Y_{i+1} [RY_i].

PROPOSITION A.2. Let X_0 be compact, X_{j+1} closed, $U_j(x_j)$ u.s.c. compact-valued point-to-set map, f_j continuous, r_j upper semicontinuous in their

domain, $j = 0, 1, \dots, i$, and $Y_{i+1}(x_i) \neq \emptyset$. Then $\text{ub}_{i+1}(\hat{x}_i, \cdot)$ and UB_{i+1} are upper semicontinuous in their domains.

This proposition was proved by Doležal [10] who generalized the proof by Boltianski [5]. The main idea is to show that the set of admissible solutions is compact. For this also another assumptions like compactness of V_j might be appropriate.

From the semicontinuity of UB_N (i.e. for $i = N$ above) it follows that the problem (1) has an optimal solution. Moreover, from the semicontinuity of the upper boundary function $\text{ub}_{i+1}(\hat{x}_i, \cdot)$ or UB_{i+1} it follows that there exists a continuous support to the sets $\hat{Y}_{i+1}(\hat{x}_i)$ or \hat{Y}_{i+1} respectively, see [38].

PROPOSITION A.3. Let f_i be a continuous and r_i an upper semicontinuous function in their domains, and let $G_i^s(x_i, x_{i+1}) \neq \emptyset$, $[U_i(x_i) \neq \emptyset, x_i \in \text{RY}_i \neq \emptyset]$ be an u.s.c. compact-valued point-to-set map [$i = N-1, N-2, \dots, 0$, $V_N \neq \emptyset$ closed]. Then $\text{rub}_i(\tilde{x}_{i+1}, \cdot)$ [RUB_i] is an upper semicontinuous function.

Let us suppose in the sequel that the functions ub_{i+1} , rub_i , UB_{i+1} , RUB_i are finite. We also introduce the following assumptions:

- (Ai) The functions r_i , g_i , h_i , f_i are locally Lipschitz continuous on W_i ,
- (Aii) The point-to-set map $G_i^s(x_i, x_{i+1}) \neq \emptyset$ is uniformly compact at $x_{i+1} \in Y_{i+1}(x_i)$ for a given $x_i \in Y_i$,
- (Aiii) The point-to-set map $G_i^s(x_i, x_{i+1}) \neq \emptyset$ is uniformly compact at $x_i \in \text{RY}_i(x_{i+1})$ for a given $x_{i+1} \in \text{RY}_{i+1}$,
- (Aiv) The point-to-set map $G_i^q(x_{i+1}) \neq \emptyset$ is uniformly compact at $x_{i+1} \in Y_{i+1}$,
- (Av) The point-to-set map $\text{RG}_i^q(x_i) \neq \emptyset$ is uniformly compact at $x_i \in \text{RY}_i$.

PROPOSITION A.4. Suppose that (Ai) and (Aii) [(Aiii)] hold and moreover that h_i , f_i are continuously differentiable. Assume that at some $u_i \in \text{O}_i^s(x_i, x_{i+1})$ [$u_i \in \text{RO}_i^s(x_{i+1}, x_i)$] $K_{ui}^0 = \emptyset$. Then $\text{ub}_{i+1}(\hat{x}_i, \cdot)$ [$\text{rub}_i(x_{i+1}, \cdot)$] is continuous at $x_{i+1} \in Y_{i+1}(x_i)$ [$x_i \in \text{RY}_i(x_{i+1})$].

PROPOSITION A.5. Suppose that (Ai), with h_i , f_i continuously differentiable, hold for $i = 0, 1, \dots, j$ and (Aiv) for $i = j$ [(Ai) with h_i , f_i continuously differentiable for $i = j, j+1, \dots, N-1$ and N when appropriate, and (Av) for $i = j$]. Assume that $K_{ui}^0 = \emptyset$ at each u_i belonging to some optimal strategy to stage j , $\mathcal{U}_j \in \mathcal{C}_j^q(x_{j+1})$ [terminating strategy from stage j , $\mathcal{H} \mathcal{U}_i \in \mathcal{H} \mathcal{C}_j^q(x_{j+1})$]. Then the function UB_{j+1} [RUB_j] is continuous at $x_{j+1} \in Y_{j+1}$ [$x_j \in \text{RY}_j$].

PROPOSITION A.6. Suppose that (Ai) and (Aii) [(Aiii)] hold. Assume that $K_{ui}^0 = \emptyset$ for all $u_i \in \text{O}_i^s(x_i, x_{i+1})$. Then $\text{ub}_{i+1}(\hat{x}_i, \cdot)$ [$\text{rub}_i(\tilde{x}_{i+1}, \cdot)$] is locally Lipschitz continuous near $x_{i+1} \in Y_{i+1}(x_i)$ [$x_i \in \text{RY}_i(x_{i+1})$].

PROPOSITION A.7. Suppose that (Ai) hold for $i = 0, 1, \dots, j$ and (Aiv) for $i = j$ [(Ai) for $i = j, j+1, \dots, N-1$ and N when appropriate, and (Av) for $i = j$]. Assume that $K_{ui}^0 = \emptyset$ at each u_i belonging to all $\mathcal{U}_j \in \mathcal{C}_j^q(x_{j+1})$ [$\mathcal{H} \mathcal{U}_j \in \mathcal{H} \mathcal{C}_j^q(x_{j+1})$]. Then the function UB_{j+1} [RUB_j] is locally Lipschitz continuous near $x_{j+1} \in Y_{j+1}$ [$x_j \in \text{RY}_j$].

PROPOSITION A.8. Suppose that (Ai) and (Aii) [(Aiii)] hold. Assume that the optimal solution u_i is unique, $K_{ui}^0 = \emptyset$ and the Lagrange multipliers $\mu_i, \lambda_i, p_{i+1}$ are unique. Then the function $\text{ub}_{i+1}(\hat{x}_i, \cdot)$ [$\text{rub}_i(\tilde{x}_{i+1}, \cdot)$] is continuously differentiable at $x_{i+1} \in Y_{i+1}(x_i)$ [$x_i \in RY_i(x_{i+1})$] and the derivative is given by:

$$\partial \text{ub}_{i+1}(\hat{x}_i, x_{i+1}) / \partial x_{i+1} = p_{i+1}^T,$$

$$[\partial \text{rub}_i(\tilde{x}_{i+1}, x_i) / \partial x_i = \partial L_i / \partial x_i],$$

where L_i is given by (A.10).

PROPOSITION A.9. Suppose that (Ai) holds for $i = 0, 1, \dots, j$ and (Aiv) for $i = j$ [(Ai) for $i = j, j+1, \dots, N-1$ and N when appropriate, (Av) for $i = j$]. Assume that the optimal strategy to stage j , $u_j \in \mathcal{U}_j^g(x_{j+1})$ is unique, $K_{ui}^0 = \emptyset$ and the Lagrange multipliers $\mu_{ij}, \lambda_{ij}, p_{i,j+1}$ are unique, $i = 0, 1, \dots, j$ [$i = j, j+1, \dots, N-1$]. Then the function UB_{j+1} [RUB_j] is continuously differentiable at $x_{j+1} \in Y_{j+1}(x_j \in RY_j)$ and the derivative is given by:

$$\partial \text{UB}_{j+1}(x_{j+1}) / \partial x_{j+1} = p_{i,j+1}^T,$$

$$[\partial \text{RUB}_j(x_j) / \partial x_j = \partial L_{jj} / \partial x_j],$$

where L_{jj} is given by (A.12).

PROPOSITION A.10. If for a fixed x_i [x_{i+1}]:

- (i) the functions r_i, g_i, h_i, f_i are k times continuously differentiable ($k \geq 2$) with respect to u on a neighbourhood of u_i ,
 - (ii) there exist Lagrange multipliers $\mu_i, \lambda_i, p_{i+1}$ such that the Karush-Kuhn-Tucker conditions (A.13) (with $\lambda_i^0 = 1$) hold,
 - (iii) $y_i^T \nabla_u^2 L_i y_i < 0$ for all $y_i \neq 0$ such that:
 - o $\nabla_u g_i^j(x_i, u_i) = 0$ for all j where $\lambda_j > 0$,
 - o (A.15a) hold,
 - (iv) the constraint qualification (A.16) holds,
 - (v) $\lambda_j > 0$ when $g_i^j(x_i, u_i) = 0$ (i.e. strict complementarity slackness),
 - (vi) the solution u_i is unique,
- then $\text{ub}_{i+1}(\hat{x}_i, \cdot)$ [$\text{rub}_i(\tilde{x}_{i+1}, \cdot)$] is k times continuously differentiable on an open neighbourhood of x_{i+1} [x_i].

PROPOSITION A.11. If for $i = 0, 1, \dots, j$ [$i = j, j+1, \dots, N-1$ and N when appropriate]:

- (i) the functions r_i, f_i, h_i, g_i are k times continuously differentiable ($k \geq 2$) on a neighbourhood of (x_i, u_i) ,
- (ii) there exist Lagrange multipliers $p_{j,i+1}, \mu_{ji}, \lambda_{ji}$ such that the Karush-Kuhn-Tucker conditions (A.14) (with $\lambda_i^0 = 1$) hold,
- (iii) $Z_i^T \nabla^2 L_i Z_i < 0$,
for all $Z_i^T = [y_i^T, z_i^T] \neq 0, Z_i \in R^{n+m}$ such that:

$$\nabla_u h_i(x_i, u_i) z_i + \nabla_x h_i(x_i, u_i) y_i = 0$$

$$\nabla_u g_i^{l_{ji}}(x_i, u_i) z_i + \nabla_x g_i^{l_{ji}}(x_i, u_i) y_i = 0 \text{ for all } l_{ji} \text{ where } \lambda_{ji}^{l_{ji}} > 0$$

$$\nabla_u g_i^l(x_i, u_i) z_i + \nabla_x g_i^l(x_i, u_i) y_i \geq 0 \text{ for all } l \in I_i$$

[for $i \neq j$ and $i \neq N$, and:

$$\nabla_u h_i(x_i, u_i) z_i = 0$$

$$\nabla_u g_i^{l_{ji}}(x_i, u_i) z_i = 0 \text{ for all } l_{ji} \text{ where } \lambda_{ji}^{l_{ji}} > 0$$

$$\nabla_u g_i^l(x_i, u_i) z_i \geq 0 \text{ for all } l \in I_i$$

for $i = j$ and $i = N$]

where I_i is given in (A.15a),

(iv) the constraint qualification (A.16) holds,

(v) $\lambda_{ji}^l > 0$ when $g_i^l(x_i, u_i) = 0$ (i.e. strict complementarity slackness),

(vi) (x_i, u_i) is the globally unique admissible solution,

then UB_{j+1} [RUB_j] is k times continuously differentiable at $x_{j+1} = f_j(x_j, u_j)$ [x_j such that $f_j(x_j, u_j) = x_{j+1}$].

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Górne ograniczenia w dyskretnym sterowaniu optymalnym z ograniczeniami

W pracy przedstawiono podstawowe konstrukcje geometryczne górnych ograniczeń w optymalnym sterowaniu dyskretnym z mieszanymi ograniczeniami na stany i sterowania. Przedyskutowano związek prezentowanego podejścia z głównymi metodami optymalizacji: zasadą maksimum, programowaniem dynamicznym i warunkami Krotowa.

Верхние ограничения в дискретном оптимальном управлении с ограничениями

В работе представлены основные геометрические конструкции верхних ограничений в оптимальном дискретном управлении со смешанными ограничениями по состоянию и управлению. Рассматривается связь представленного подхода с основными методами оптимизации: принципом максимума, динамическим программированием и условиями Кротова.