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On a norm scalarization in infinite dimensional Banach spaces

by

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A norm scalarisation was studied for finite dimensional Euclidean space by Savlukadze [1, 2]. His results were extended partially for Hilbert space by Wierzbicki [3]. In the present note a norm scalarisation for Banach spaces is investigated.

Let E be a linear space. Let D be a convex cone in E. Let Q be a set in E. A point $p \in Q$ is called D-optimal if

$$(p-D) \cap Q=\{p\}. \tag{1}$$

Savlukadze [1, 2] has proved that if $E=R^n$ and $D=\{x=(x_1, ..., x_n),\ x_i \geqslant 0,\ i=1, 2, ..., n\}$ then we can find a D-optimal point in a following way.

Let

$$y_i=\inf \{x_i : (x_1, \ldots, x_n) \in Q\}.$$

The point $y=(y_1, ..., y_n)$ is called "utopia point", since in general y is not necessary belonging to Q.

Now let $x \in Q$ be such a point that

$$\rho(y, x)=\inf_{z \in Q} \rho(y, z)$$

where ρ is the Euclidean metric in R^n. Such x exist, provided Q is closed, because $Q \subset R^n$. Savlukadze [1, 2] has proved that x is a D-optimal point.

For infinite dimensional Hilbert space H a result of similar character was given by Wierzbicki [3]. Namely, Wierzbicki take an arbitrary point $p \in Q$. Let $\Gamma_p=Q \cap (p-D)$. Let x be such a point that

$$\rho(x, p)=\sup_{z \in \Gamma_p} \rho(x, z),$$

where ρ is a Hilbert distance in H. Wierzbicki proved that x is D-optimal, provided

$$D \subset D^*. \tag{2}$$

In the present note we shall extend the results of Savlukadze and Wierzbicki for infinite dimensional Banach spaces.

Let E be a Banach space. We assume that the cone D satisfies a following condition

$$D \cap (x-D) \subset K_{\|x\|}(0) \cup \{x\} \quad \text{for all} \quad x \in E \tag{3}$$

where

$$K_r(q) = \{z: \|z-q\| < r).$$

THEOREM 1. Let E be a Banach space. Let D be closed and satisfies (3). Let Q be an arbitrary closed set in E. Let p be a point, such that

$$Q \subset p+D. \tag{4}$$

Let $x_0 \in Q$ be a point, such that

$$\|x_0-p\| = \inf\{\|z-p\|: z \in Q\}. \tag{5}$$

Then x_0 is D-optimal.

Proof. By condition (4) $x_0 \in p+D$. Thus by (3).

$$(p+D) \cap (x_0-D) \subset K_{\|x_0-p\|}(p) + \{x_0\}.$$

By definition of x_0', $K_{\|x_0-p\|}(p) \cap Q = \emptyset$, Therefore

$$x_0 - D \cap Q = x_0 - D \cap p + D \cap Q = \emptyset. \qquad \text{Q.E.D.}$$

THEOREM 2. Let E be a Banach space. Let D be a closed cone satisfying (3). Let q be an arbitrary point of Q. Let $\Gamma_q = (q-D) \cap Q$. Let $x_0 \in Q$ be a point satisfying

$$\|x_0-q\| = \sup\{\|z-q\|: z \in \Gamma_q\}.$$

Then x_0 is D-optimal.

Proof. By the symmetry of balls in Banach spaces from (3) we get

$$(-D) \cap (x+D) \subset K_{\|x_0\|}(0) \cup \{x_0\}. \qquad \text{Q.E.D.} \tag{3'}$$

Hence

$$(q-D) \cap (x_0+D) \subset K_{\|x_0\|}(q) \cup \{x_0\}. \tag{6}$$

Since $K_{\|x_0-q\|}(q)$ is a convex open set, the x belongs to boundary of this set, D is closed, (6) implies that

$$(x_0-D) \cap K_{\|x_0-q\|}(q) = \{x_0\}. \tag{7}$$

Thus

$$(x_0-D) \cap \Gamma_q = \{x_0\}. \tag{8}$$

Since $x_0 - D \subset q - D$ by (8) we get

$$(x_0-D) \cap Q = (x_0-D) \cap (q-D) \cap Q = (x_0-D) \cap \Gamma_q = \{x_0\}. \qquad \text{Q.E.D.} \tag{9}$$

Therefore x_0 is D-optimal.

Now we shall show relation between condition (3) and condition (2) given by Wierzbicki.

THEOREM 3. Let E be a Hilbert space. Then (2) and (3) are equivalent.

Proof. (2)→(3). By definition of D^*, if $x \in D$ and $x^* \in D^*$, $(x^*, x) \geqslant 0$. Thus for $y \in -D^*$, $(y, x) \leqslant 0$. It implies, that the angle between y and x is not smaller then $\pi/2$.

Thus everything can be reducted to a two dimensional consideration. Let $q \in \in \operatorname{lin}(x, y)$. Since between $x \in D$ and $y \in -D^*$ the angle is not smaller than $\pi/2$, thus the lines $\{tx\}$ $\{q - sy\}$, t, s being reals, must intersect inside the ball $K_{\|q\|}(0)$. It implies

$$D \cap (q - D^*) \subset K_{\|(q)\|}(0) \cup \{q\}. \tag{10}$$

Thus by (2) we trivially get (3).

(3)→(2). Suppose that (2) does not hold. Then here are $x, y \in D$ such that $(x, y) < 0$. Let

$$q_\alpha = x + \alpha y, \quad 0 < \alpha < 1.$$

It is easy to verify that

$$x = q_\alpha - \alpha y \in q_\alpha - D.$$

On the other hand

$$\|q\|^2 = (x + \alpha y, x + \alpha y) = \|x\|^2 - 2\alpha (x, y) + \alpha^2 \|y\|^2$$

and for sufficiently small α

$$\|x\|^2 < \|q_\alpha\|^2.$$

It implies that $D \cap (q_\alpha - D)$ is not contained in $K_{\|q\|}(0) \cup \{q_\alpha\}$. Hence (3) does not hold.

In many cases there is no such a point p that (4) holds. It can follows from fact that, either Q is not bounded, or D does not have interior.

For these reason a following obvious extensions of Theorem 1 are important.

THEOREM 1′. Let E be a Banach space, D be a closed cone. Q be a closed set. Let p be an arbitrary point belonging to E. Let $x_0 \in Q$ be a point satisfying (5).

If $x_0 \in p + D$ and

$$(p + D) \cap (x_0 - D) \subset K_{\|x_0 - p\|}(p) \cup \{x_0\}$$

then x_0 is D-optimal.

THEOREM 1″. Let E be a Banach space, D and D_1 be closed cones, $D \subset D_1$. Let Q be a closed set contained in $p + D_1$. Let $x_0 \in Q$ be a point satisfying (5).

If

$$(p + D_1) \cap (x_0 - D) \subset K_{\|x_0 - p\|}(p) \cup \{x_0\}$$

then x_0 is D-optimal.

Since condition (4) plays an important role, we are interested how is the set of those p that (4) holds.

THEOREM 4. Let E be a linear space. Let D be a convex cone. Let Q be an arbitrary set. Then the set

$$p=\{p \in E: Q \subset p+D\} \tag{11}$$

is a convex set.

Proof. Let $p, q \in P$. Let z be an arbitrary element of Q. By the definition of P we can represent z in the foms

$$z=p+x=q+y \tag{12}$$

where $x, y \in D$. Then for $\alpha, \beta \geqslant 0$, $\alpha+\beta=1$

$$(\alpha+\beta) z=\alpha p+\beta q+\alpha x+\beta y. \tag{13}$$

(13) implies that $z \in \alpha p+\beta q+D$, since $\alpha x+\beta y \in D$. Therefore

$$Q \subset \alpha p+\beta q+D. \qquad \text{Q.E.D.} \tag{14}$$

Hence P is convex.

THEOREM 5. Let E be a Banach space. Let D be a closed convex cone in E. Let Q be a closed set. Let P be a following set (11)

$$P=\{p \in E: Q \subset p+D\}.$$

Then the set P is closed.

Proof. Let $\{p_n\}$ be a sequence of elements of P convergent to $p \in E$. Let z be an arbitrary element of Q. By the definition of P, z can be represented by a following form:

$$z=p_n+x_n, \tag{15}$$

where $x_n \in D$.

Since $\{p_n\}$ is a convergent sequence, $\{x_n\}$ is convergent too. Let $x=\lim x_n$. Since D is closed, $x \in D$. By (15) $z=p+x$. If implies that

$$Q \subset p+D. \qquad \text{Q.E.D.}$$

By definition of P, $p \in P$, and P is closed.

References

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O skalaryzacji normowej w nieskończenie wymiarowych przestrzeniach Banacha

W pracach [1, 2] Savlukadze podał metodę skalaryzacji normowej dla przestrzeni euklidesowej skończenie wymiarowej. Rezultaty jego były częściowo uogólnione przez Wierzbickiego [3] dla nieskończenie wymiarowej przestrzeni Hilberta. W niniejszej nocie rozszerzone zostały wyniki Wierzbickiego o skalaryzacji normowej na przypadek nieskończenie wymiarowej przestrzeni Banacha.

Скаляризация нормы в бесконечном банаховом пространстве

Скаляризация нормы для конечномерного евклидового пространства изучалась в работах Салюковадзе [1], [2]. Эти результаты были частично расширены Вежбицким для гильбертового пространства [3]. В данной работе исследовалась скаляризация нормы для банахового пространства.